Directed graphs and combinatorial properties of groups and semigroups.

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Abstract

This thesis is a study of several combinatorial properties of groups and semigroups. Research on combinatorial properties on groups and semigroups with all infinite subsets having certain unavoidable regularities originates from Ramsey theory and has been the subject of active investigations. In the thesis we explore several properties defined in terms of directed graphs.

Chapter 1 outlines some of the earlier achievements in this area and sets the context for the thesis. The thesis continues the work by a number of well-known authors, and contains new results that give rise to interesting connections between graph, group and semigroup theoretic methods.

Preliminaries and background information are included in Chapter 2 for convenience of the reader. Several technical facts used in proofs are also referenced.

Chapter 3 deals with a combinatorial property related to power graphs. The *power graph* of a semigroup S is a directed graph with the set S of vertices, and with all edges (u, v) such that $u \neq v$ and v is the power of u. We introduce a combinatorial property defined for power graphs by analogy with several properties considered earlier. The first main theorem completely describes all pairs (S, D), where S is a semigroup and D is a directed graph, and S satisfies this combinatorial property with respect to D. The structure of the power graphs of all finite abelian groups is then described.

Chapter 4 is devoted to Cayley graphs. Let T be a subset of a semigroup S. The *Cayley graph* $\mathrm{Cay}(S,T)$ of S with respect to T is defined as the graph with the set S of vertices and with all edges (x, y), where $x \neq y$ and $xt = y$ for some $t \in T$. Cayley graphs play important roles in combinatorial group and semigroup theory. For each finite directed graph D, we obtain conditions necessary and sufficient for the Cayley graph $\mathrm{Cay}(S, S)$ of a semigroup S to contain a subgraph isomorphic to D. The second main theorem of this chapter shows that every infinite semigroup S has an infinite subset T inducing a null subgraph in the Cayley graph $\mathrm{Cay}(S,T)$. A natural question that arises is when the Cayley graph of a semigroup belongs to one of the classes well known in graph theory. The next theorems in this chapter characterise all finite inverse semigroups and all commutative inverse semigroups with bipartite Cayley graphs. We then obtain necessary and sufficient conditions for the Cayley graph of a semigroup to comprise the disjoint union of complete graphs. This result is used to describe all monoids S and subsets T of S such that $\mathrm{Cay}(S,T)$ is isomorphic to the disjoint union of complete graphs.

The *divisibility graph* of a semigroup S has edges (u, v), where u belongs to the ideal generated by v. The main theorems of Chapter 5, for each directed graph D, characterise all commutative and completely 0-simple semigroups with all infinite subsets containing divisibility subgraphs isomorphic to D. A description is also given for all monomial matrix semigroups which possess the same property.

Chapter 6 examines the concept of an annihilator set. Annihilators have been studied in combinatorial semigroup theory, in particular, in relation to unavoidable regularities in infinite sequences of elements. The *annihilator graph* of a semigroup S has edges (u, v) whenever $uv = 0$ and $u \neq v$. The main theorems of this chapter describe all commutative and linear semigroups S, where every infinite subset of S induces an annihilator graph that has a subgraph isomorphic to a finite directed graph D.

The author has also obtained and published new results on other related topics. A few of his theorems on automata, their languages and syntactic monoids have been included in Appendix 1.

The results of this thesis have been published in [31], [32], [34] and [35].

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Chapter 1

Introduction

In 1930, F. P. Ramsey [47] published his now classical paper, "On a problem of formal logic", which dealt with decidability questions in logic. In loose terms Ramsey showed that when a sufficiently large class of objects is partitioned into a finite number of classes, at least one class contains a certain property or substructure. Applications of Ramsey's Theory exist in many fields of mathematics, including set theory, probability theory, analysis and graph theory (see [19]).

Ramsey's Theorem is also useful when dealing with unavoidable regularities in infinite groups and semigroups. For example, Justin [24] introduced the concept of a repetitive semigroup and developed a theory, describing all commutative semigroups which are repetitive. Linear semigroups which are repetitive have been described by Kelarev and Shumyatsky in [36]. Many other investigations arising from this concept have followed (see [17], [18] and [25]).

B. H. Neumann [41] proved by using Ramsey's Theorem that the class of all infinite groups which are centre-by-finite possess the combinatorial property that every infinite sequence of elements contains a pair of elements that commute. This fact underpins the results in Chapter 2.

This thesis is devoted to the investigation of several combinatorial properties of semigroups, where the associated graph possesses some type of unavoidable regularity. We say that a semigroup S is D-saturated for a particular combinatorial property if every infinite subset of S induces a graph that contains D as a subgraph. Each chapter concentrates on a different combinatorial property and structural theorems are given that describe various classes of semigroups satisfying that particular combinatorial property.

Chapter 2 provides most of the background information and known facts needed for subsequent chapters. All preliminaries are found in this chapter, unless a result used is miscellaneous and local to another chapter.

In Chapter 3, we describe all pairs (S, D), where S is a semigroup, D is a finite graph and S is power D-saturated. Necessary and sufficient conditions are first given for the case when S is a group. This result is used to characterise all D-saturated commutative and linear semigroups, which in turn allows a description of D-saturated semigroups in general. For any abelian group G, the power graph of G is described.

Cayley graphs of groups are significant not only in group theory, but also in constructions of interesting graphs with nice properties (see [44]). They have been the subject of many investigations (see, for example [5], [7], and [16]). The concept of the Cayley graph of a semigroup was introduced by Bohdan Zelinka [57], and it is natural to expect that the more general concept can be used to define various types of graphs with new combinatorial properties. For example, vertex transitive Cayley graphs have been characterised in [28] as well as undirected Cayley graphs (see [26]). This concept is also important for combinatorial semigroup theory (see [40] and [53]).

Chapter 4 begins by describing all pairs (D, S), where D is finite with at least one edge and S is Cayley D-saturated with respect to S. We then show that for a finite graph with edges there are no infinite semigroups S which are Cayley D-saturated with respect to all subsets T of S. A natural question that arises is when the Cayley graph of a semigroup belongs to one of the classes well known in graph theory. We describe all finite inverse semigroups with bipartite Cayley graphs. All commutative inverse semigroups with bipartite Cayley graphs are then characterised. We then obtain necessary and sufficient conditions for all semigroups whose Cayley graphs are isomorphic to the disjoint union of complete graphs. This result is used to describe all pairs (S, T), where S a monoid, T is a subset of S and $\mathrm{Cay}(S, T)$ is isomorphic to the disjoint union of complete graphs.

Chapter 5 investigates semigroups which are divisibility D-saturated. A complete description of all commutative divisibility D-saturated semigroups is obtained. Corresponding conditions are then given for completely 0-simple semigroups and this result is used to describe all monomial matrix semigroups which satisfy the same combinatorial property.

The concept of an annihilator set has been considered in combinatorial semigroup theory, graph theory and the study of formal languages (see [13], [42] and [56]). For example, Justin and Kelarev [25] showed that annihilators are connected with unavoidable regularities in infinite sequences of elements in semigroups. A natural question that arises concerns infinite semigroups which possess other combinatorial properties, defined in terms of annihilator sets and their associated graphs. Chapter 6 examines the class of all annihilator D-saturated semigroups. Necessary and sufficient conditions are given first for all commutative semigroups which are annihilator D-saturated, and then for semigroups S, where $S/\mathcal{J}$ is finite. We then describe the structure of all linear semigroups which satisfy this combinatorial property.

For each finite undirected graph the graph algebra associated with that graph is a set of objects, together with a binary operation. Graph algebras make it possible to apply methods of universal algebras to various problems of discrete mathematics and computer science. They have been investigated by several authors (see [43] and [46] for references). An algorithmic description is given of all regular languages recognised by graph algebras of finite graphs in Appendix 1.

Any lemmas which are not original have been referenced. All other results have been obtained either by myself or in collaboration with co-authors of refereed papers.

Chapter 2

Preliminaries

Groups

A *binary operation* R on a set X is a map $R : X \times X \to X$. A *group* is a set G equipped with a binary operation " $\circ$ " such that:

(i) G is closed under $\circ$;

(ii) $\circ$ is *associative*, that is $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in G$;

(iii) there exists an element $e \in G$ such that $g \circ e = g$, for all $g \in G$; and

(iv) to each element $g \in G$ there corresponds an element $h \in G$ such that $g \circ h = e$.

It follows from (iii) and (iv) that $e \circ g = g$, for all $g \in G$ and that $h \circ g = e$ (see [50], Lemma 1.1.2). The element e is called the *identity* of G. For a group G with element $g \in G$, the identity of G is sometimes also denoted by e_g.

If $x \circ y = y \circ x$, for all $x, y \in G$, then G is said to be *abelian.*

It is customary to identify the group $(G, \circ)$ and its underlying set G provided there is no possibility of confusion as to the intended group operation.

A subset H of a group G is a *subgroup* if $(H, \circ)$ is a group, where $\circ$ is the group operation restricted to H.

Lemma 1 ([50], Proposition 1.3.1) *A subset H of a group $(G, \circ)$ is a subgroup of G if and only if H is non-empty and $xy^{-1} \in H$ whenever $x, y \in H$.*

The *order* of a group G is defined to be the cardinality of the underlying set G and is denoted by $|G|$. The order of an element g is the least positive integer n such that $g^n = e$, if n exists. In this case g is said to have finite order. Otherwise, g has infinite order. For a prime p, a finite group is called a *p-group* if its order is a power of p.

Let G be an abelian group. If p is a prime, then the elements with order of some power of p form a subgroup G_p called the *p-primary component* of G. The *cyclic group* of order p is denoted by $\mathbb{Z}_p$, and $\mathbb{Z}_{p^\infty}$ stands for the *quasicyclic p-group*, that is, the infinite group with generators $g_1, g_2, \ldots$ such that $g_1^p = e_{g_1}$ and $g_i^p = g_{i-1}$, for all $i > 1$.

Let G be an abelian group. A subset H of G is *independent* if, given elements $h_1, h_2, \ldots h_r$ of H and integers $m_1, m_2, \ldots, m_r$, then

$$m_1h_1 + m_2h_2 + \ldots + m_rh_r = 0 \text{ implies that } m_ih_i = 0, \text{ for all } i.$$

If p is prime and G abelian, then the *p-rank* of G is defined as the cardinality of a maximal independent subset of elements of p-power order.

A *torsion group* (or *periodic group*) is a group all of whose elements have finite order. The *centre* $C(G)$ of a group G is the subset of those elements which commute with every element in G.

The *right* (resp., *left*) *cosets* of a subgroup H of a group G are the sets $\{Hg : g \in G\}$ (resp., $\{gH : g \in G\}$). The cardinalities of the left and right cosets coincide and are called the *index* of H in G. A subgroup H of G is *normal* if $gH = Hg$, for all $g \in G$. If N is a normal subgroup of a group G, the *quotient group* of N in G is denoted by G/N and is the set of all cosets of N in G equipped with the group operation $(Ng_1)(Ng_2) = N(g_1g_2)$.

A group G is *centre-by-finite* if the quotient group $|G/C(G)|$ is finite.

The *direct product* $G_1 \times G_2 \times \ldots G_n$ of a finite set of groups is the group with the set

$$\{(g_1, \ldots, g_n) \mid g_i \in G_i \text{ for } 1 \leq i \leq n\},$$

and multiplication defined by $(g_1, \ldots, g_n)(h_1, \ldots, h_n) = (g_1h_1, \ldots, g_nh_n)$. The projection π_i is a homomorphism $\pi_i : G \to G_i$ such that $\pi_i(g_1, \ldots, g_n) = g_i$.

For other group theoretic terminology the reader is referred to [50].

Semigroups

A *semigroup* $(S, \cdot)$ is a set S equipped with an associative binary operation "$\cdot$". The abbreviation S is used instead of $(S, \cdot)$ throughout the thesis. A non-empty subset T of a semigroup S is a *subsemigroup* if T is closed under "$\cdot$". If T is a subset of a semigroup S, then $\langle T \rangle$ denotes the subsemigroup of S generated by T.

A semigroup S with zero 0 adjoined is denoted by S^0 and is defined by

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero,} \\ S \cup \{0\} & \text{otherwise.} \end{cases}$$

By analogy with the case of S^0, the *monoid* S^1 is defined by

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

An element x in a monoid S is called a *unit* if there exists $y \in S$ such that $xy = yx = 1$. The set of all units of S forms a group called the *group of units* ([12], Theorem 1.10).

A semigroup is *null* if the product of any two elements is zero. An *idempotent* in a semigroup is any element e, for which $e^2 = e$.

A *binary relation* on a set X is a subset ρ of the cartesion product $X \times X$. In particular, the empty set $\emptyset$ of $X \times X$ is included in the set of all the binary relations on $X \times X$ as well as the *equality* relation

$$1_X = \{(x, x) \mid x \in X\}.$$

The *converse* ρ^{-1} of ρ is defined by

$$\rho^{-1} = \{(x, y) \in X \times X \mid (y, x) \in \rho\}.$$

A *relation* ρ on a set X is

reflexive	if and only if	$1_X \subseteq \rho$;
symmetric	if and only if	$\rho = \rho^{-1}$; and
transitive	if and only if	$\rho \circ \rho = \rho$.

An *equivalence relation* is a reflexive, symmetric and transitive relation.

For a reflexive relation ρ on X, the *transitive closure* of ρ is defined by

$$\rho^\infty = \bigcup\{\rho^n \mid n \geq 1\}.$$

For an arbitrary relation ρ on X, the smallest equivalence *generated* by ρ (see [21], Proposition 1.4.9) is given by

$$\rho^e = [\rho \cup \rho^{-1} \cup 1_X]^\infty.$$

An equivalence relation ρ on a semigroup is a *congruence* if

$$(\forall s, t, s', t' \in S) \quad (s,t) \in \rho \text{ and } (s',t') \in \rho \quad \Rightarrow (ss', tt') \in \rho.$$

For an arbitrary binary relation ρ on a semigroup S, $\rho^\#$ denotes the smallest congruence on S containing ρ. It is given by $\rho^\# = (\rho^c)^e$ (see [21], Proposition 1.5.8), where

$$\rho^c = \{(xay, xby) \mid x, y \in S^1, (a,b) \in \rho\}.$$

If $c, d \in S$ are such that $c = xay, d = xby$, for some $x, y \in S^1$, where either (a,b) or (b,a) belongs to a relation ρ, then c is connected to d by an *elementary ρ-transition.* This gives

Lemma 2 ([21], Proposition 1.5.9) *Let ρ be a relation on a semigroup S, and let $a, b \in S$. Then $(a,b) \in \rho^\#$ if and only if either $a = b$ or, for some $n \in \mathbb{Z}^+$, there is a sequence*

$$a = z_1 \to z_2 \to \ldots \to z_n = b$$

of elementary ρ-transitions connecting a to b.

A binary relation ρ on a set X is called a *partial order* if

$(x,x) \in \rho$, for all $x \in X$,	i.e., ρ is reflexive;
$(\forall x, y \in X)\ (x,y), (y,x) \in \rho \Rightarrow x = y$,	i.e., ρ is *antisymmetric*;
$(\forall x, y, z \in X)\ (x,y), (y,z) \in \rho \Rightarrow (x,z) \in \rho$,	i.e., ρ is transitive.

Alternatively, if $(x,y) \in \rho$ then we can write $x \leq y$. A *partially ordered set* is a system consisting of a non-empty set X and a partial order $\leq$.

For a non-empty subset Y of a partially ordered set $(X, \leq)$, an element a of Y is *minimal* if there is no element $y \in Y$ such that $y \leq a$. An element

$c \in X$ is a *lower bound* of Y if $c \leq y$ for every $y \in Y$. *Maximal* elements and *upper bounds* are defined analogously. If the set of lower (resp., upper) bounds is non-empty and has a maximum (resp., minimum) element d, then d is said to be the *meet* (resp., *join*) of Y. If the meet (resp., join) of every pair of elements of $(X, \leq)$ (resp., $(X, \geq)$) exists, then $(X, \leq)$ (resp., $(X, \geq)$) is said to be a *lower semilattice* (resp., *upper semilattice*). The term *semilattice* will be used to mean lower semilattice.

Two elements x, y in a partial order (X, ρ) are *comparable* if either $x \leq y$ or $y \leq x$. Otherwise they are *incomparable*. A subset of a partially ordered set with the property that every pair of elements is comparable is called a *chain*. A subset of a partially ordered set is called an *antichain* if all of its elements are incomparable (see [52], §11.8).

A binary relation ρ from A to B (that is, a subset of $A \times B$) is a *map* from A to B if, for every $a \in A$, there is a unique $b \in B$ such that $(a, b) \in \rho$. The map $\rho : A \to B$ is said to be *injective* if

$$(\forall a_1, a_2 \in A)\; \rho(a_1) = \rho(a_2) \Rightarrow a_1 = a_2.$$

The map $\rho : A \to B$ is said to be *surjective* if

$$(\forall b \in B)(\exists a \in A)\; \rho(a) = b.$$

A map $\rho : A \to B$ is a *bijection* if it is both injective and surjective, that is, if

$$(\forall b \in B)(\exists! a \in A)\; \rho(a) = b.$$

A map $\rho : A \to B$, where $(S, .)$ and $(T, .)$ are semigroups, is called a *morphism* if, for all $x, y \in S$,

$$\rho(xy) = \rho(x)\rho(y).$$

Definition 3 ([22], §1.5) Given two monoids S and T, it is said that S *divides* T if there exists a submonoid U of T and a surjective morphism $\phi : U \to S$.

If $a, b \in S$, then $a\mathcal{L}b$ (resp., $a\mathcal{R}b, a\mathcal{J}b$) if and only if $S^1a = S^1b$ (resp., $aS^1 = bS^1$, $S^1aS^1 = S^1bS^1$). The relation $\mathcal{H}$ (resp., $\mathcal{D}$) is defined as the intersection (resp., join) of $\mathcal{L}$ and $\mathcal{R}$. These equivalence relations are known as *Green's equivalences* (see [21] §2.1). The $\mathcal{L}$-class (resp., $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class, $\mathcal{J}$-class) containing the element a is denoted by L_a (resp., R_a, H_a, D_a, J_a).

If T is a subsemigroup of S, then superscripts are used in order to avoid ambiguity about the meaning of a particular equivalence relation. For example,

$a\mathcal{L}^S b$ means that there exist $u, v \in S^1$ such that $a = ub, b = va$, while $a\mathcal{L}^T b$ means that there exist $w, x \in T^1$ such that $wa = b$ and $xb = u$. For a subsemigroup T of S, the notations

$$L_a^T = \{t \in T \mid (a,t) \in \mathcal{L}^T\} \text{ and } L_a^S = \{s \in S \mid (a,s) \in \mathcal{L}^S\}$$

are also used to avoid ambiguity.

The next lemma is used frequently throughout the thesis.

Lemma 4 ([21], Proposition 2.3.7) *Let a, b be elements in a $\mathcal{D}$-class D. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.*

A non-empty subset I of S is a *left* (resp., *right, two-sided*) *ideal* of S if $SI \subseteq I$ (resp., $IS \subseteq I$, both $SI \subseteq I$ and $IS \subseteq I$). If S has a minimal two-sided ideal K, then K is called the *kernel* of S.

A semigroup S is called *simple* (resp., *right simple*) if S is the only ideal (resp., right ideal) of S.

For an element $a \in S$, the *principal left* (resp., *right, two sided ideal*) *generated by* a is the set S^1a (resp., aS^1, S^1aS^1) and is denoted by $L(a)$ (resp., $R(a), J(a)$).

Since $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are defined in terms of ideals, the inclusion order among these ideals induces a partial order among the equivalence classes:

$$\begin{aligned} L_a \leq L_b \quad &\text{if} \quad S^1a \subseteq S^1b; \\ R_a \leq R_b \quad &\text{if} \quad aS^1 \subseteq bS^1; \\ J_a \leq J_b \quad &\text{if} \quad S^1aS^1 \subseteq S^1bS^1. \end{aligned}$$

Thus $S/\mathcal{L}, S/\mathcal{R}$ and $S/\mathcal{J}$ may be regarded as partially ordered sets.

An element a of a semigroup S is *regular* if there exists $x \in S$ such that $axa = a$. A semigroup is *regular* if all its elements are regular. An element a' is the *inverse* of an element a if $aa'a = a$ and $a'aa' = a'$. An *inverse semigroup* is a regular semigroup such that every element has a unique inverse. The set of idempotents in an inverse semigroup forms a semilattice with respect to a *natural partial order*, defined by $e \leq f$ if and only if $ef = fe = e$.

Suppose that G is a group, I and Λ are non-empty sets, and $P = [p_{\lambda i}]$ is a $(\Lambda \times I)$-matrix with entries $p_{\lambda i} \in G$, for all $\lambda \in \Lambda$, $i \in I$. The *Rees matrix* *semigroup* $\mathcal{M}(G; I, \Lambda; P)$ over G with *sandwich-matrix* P consists of all triples $(g; i, \lambda)$, where $i \in I$, $\lambda \in \Lambda$, and $g \in G$, with multiplication defined by the rule

$$(g_1; i_1, \lambda_1)(g_2; i_2, \lambda_2) = (g_1 p_{\lambda_1 i_2} g_2; i_1, \lambda_2).$$

A semigroup is said to be *completely simple* if it has no proper ideals and has a minimal idempotent with respect to the natural partial order. Similarly, a semigroup with zero is *completely 0-simple* if it has no proper nonzero ideals and has a minimal nonzero idempotent. It is well known that every completely simple semigroup is isomorphic to a Rees matrix semigroup $\mathcal{M}(G; I, \Lambda; P)$ over a group G (see [21], Theorem 3.3.1), and every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ over a group G with zero adjoined. Conversely, every semigroup $\mathcal{M}(G; I, \Lambda; P)$ is completely simple, and a semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ is completely 0-simple if and only if each row and column of P contains at least one nonzero entry (see [21], Theorem 3.2.3).

Let G be a group, $M = \mathcal{M}(G; I, \Lambda; P)$, and let $i \in I$, $\lambda \in \Lambda$. Put

$$\begin{aligned} G_{*\lambda} &= \{(g; i, \lambda) \mid g \in G, i \in I\}, \\ G_{i*} &= \{(g; i, \lambda) \mid g \in G, \lambda \in \Lambda\}, \\ G_{i\lambda} &= \{(g; i, \lambda) \mid g \in G\}. \end{aligned}$$

In the case where $M = \mathcal{M}^0(G; I, \Lambda; P)$ the zero is included in all of these sets. That is,

$$\begin{aligned} G_{*\lambda} &= \{0\} \cup \{(g; i, \lambda) \mid g \in G, i \in I\}, \\ G_{i*} &= \{0\} \cup \{(g; i, \lambda) \mid g \in G, \lambda \in \Lambda\}, \\ G_{i\lambda} &= \{0\} \cup \{(g; i, \lambda) \mid g \in G\}. \end{aligned}$$

The following facts are well known and are collected in a separate lemma.

Lemma 5 *Let G be a group, and let $M = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup. Then for all $i, j \in I$, $\lambda, \mu \in \Lambda$,*

(i) *the set $G_{*\lambda}$ is an $\mathcal{L}$-class of M and a minimal nonzero left ideal of M;*

(ii) *the set G_{i*} is an $\mathcal{R}$-class of M and a minimal nonzero right ideal of M;*

(iii) *the set $G_{i\lambda} \setminus \{0\}$ is an $\mathcal{H}$-class of M, and $G_{i\lambda}$ is a left ideal of G_{i*} and a right ideal of $G_{*\lambda}$;*

(iv) $|G_{i\lambda}|=|G_{j\mu}|$;

(v) *each maximal subgroup of* M *coincides with* $G_{j\mu} \setminus \{0\}$, *for some* $j \in I$, $\mu \in \Lambda$;

(vi) *if* $p_{\lambda i} \neq 0$, *then* $G_{i\lambda} \setminus \{0\}$ *is a maximal subgroup of* M *isomorphic to* G;

(vii) *if* $p_{\lambda i} = 0$, *then* $G_{i\lambda}^2 = 0$;

(viii) *every* $\mathcal{L}$*-class of* M *contains at least one maximal subgroup,* $G_{j\mu}$;

(ix) *every* $\mathcal{R}$*-class of* M *contains at least one maximal subgroup,* $G_{j\mu}$.

The *quotient semigroup* S/ρ of a semigroup S by a congruence ρ is the set of all equivalence classes of ρ with a binary operation $(\rho a)(\rho b) = \rho(ab)$. For a congruence ρ, the natural map $\rho^\natural : S \to S/\rho$ is defined by $\rho^\natural s = \rho s$, for all $s \in S$.

If I is an ideal of a semigroup, then the *Rees quotient semigroup* is the semigroup with zero obtained by identifying with 0 all elements of the ideal I.

If I, J are ideals of a semigroup S and $J \subset I$, then the Rees quotient semigroup I/J is called a *factor* of S. In the case where $J = \emptyset$, we put $I/J = I$.

A *finite ideal series* is a chain of ideals

$$\emptyset = S_0 \subset S_1 \subset \ldots \subset S_n = S.$$

A *principal series* of a semigroup S is a finite maximal chain

$$\emptyset = S_0 \subset S_1 \subset \ldots \subset S_n = S$$

of ideals of $S_i (i = 0, 1, \ldots n)$. The *factors* of the principal series are the Rees quotient semigroups $S_i/S_{i-1} (i = 1, 2, \ldots n)$. Each factor is either simple, 0-simple or null (see [12], Corollary 2.39). If S admits a principal series and each factor is either simple or 0-simple, then S is said to be *semisimple.*

A commutative semigroup Y is a semilattice if every element of Y is an idempotent. A semigroup S is said to be a *semilattice* Y *of its subsemigroups* S_y, where $y \in Y$, if $S = \cup_{y \in Y} S_y$ is a disjoint union of its subsemigroups S_y, and $S_x S_y \subseteq S_{xy}$, for all $x, y \in Y$. A semigroup is *Archimedean* if, for each pair $a, b \in S$, there exists a positive integer n such that $a^n \in bS^1$. Every commutative semigroup is uniquely represented as a semilattice Y of Archimedean semigroups S_y (see [20], Theorem 4.2.2). The S_y are called the *Archimedean components* of S. The largest subgroup of S_y will be denoted by G_y. If S_y has no idempotents, then $G_y = \emptyset$. If an Archimedean component S_y has an idempotent, it will be denoted by e_y. Then the ideal $G_y = e_y S_y$ is a group, and the quotient semigroup S_y/G_y is nil (see [20], Proposition 4.2.3).

An *ideal extension* of a semigroup S by a semigroup Q is a semigroup E such that S is an ideal of E and the Rees quotient E/S is isomorphic to Q.

An element s of S is said to be *periodic* if there exist positive integers m, n such that $s^{m+n} = s^m$. A semigroup S is *periodic* if all elements of S are periodic.

An *epigroup* is a semigroup such that a power of each element belongs to a subgroup. The class of all epigroups contains a wide variety of semigroups. In particular all periodic and completely 0-simple semigroups are epigroups (see [52], §1.1). A subsemigroup of an epigroup, which is itself an epigroup is called a *subepigroup*.

Lemma 6 ([52], Proposition 1.3.3) *The group G_e is the greatest subgroup contained in K_e, coincides with the set eK_e and is an ideal of the subepigroup $\langle K_e \rangle$.*

A semigroup is *nilpotent* if $S^n = 0$, for some natural number n. An element s is *nil* if $s^n = 0$, for some n. If all elements of S are nil then S is called a *nilsemigroup*.

A *left zero semigroup* S satisfies the identity $ab = a$, for all $a, b \in S$.

For a *skew field* K (see [55] §3.1), the set of all $n \times n$ matrices with entries in K is denoted by $M_n(K)$. A subsemigroup of $M_n(K)$ is called a *linear semigroup*. For an element $s \in M_n(K)$, the *rank* of s means the rank of the matrix s.

A matrix is said to be *monomial* if every row and column contains at most one nonzero entry. If G is a group, then the set of all $n \times n$ monomial matrices over $G^0 = G \cup \{0\}$ forms a semigroup and is denoted by $M_n(G)$ (see [23]). A *matrix semigroup* is a subsemigroup of $M_n(K)$ or $M_n(G)$, for some n, K and G. The group of $j \times j$ invertible matrices over a field K (see [49], §1.2) is denoted by $GL_j(K)$.

The *direct product* $S \times T$ of semigroups S and T is the set

$$\{(s,t) \mid s \in S, t \in T\}$$

with multiplication defined by $(s,t)(s',t') = (ss',tt')$.

An *alphabet* is a finite non-empty set A whose elements are *letters*. A *word* is a finite sequence $(a_1a_2\dots a_n)$ of letters of A. The set of all finite non-empty words will be denoted by A^+. A binary operation is defined on A^+ by concatenation:

$$(a_1a_2\dots a_n)(b_1b_2\dots b_m) = (a_1a_2\dots a_nb_1b_2\dots b_m).$$

With respect to this operation, A^+ is a semigroup, called the *free semigroup*. If we adjoin an identity 1 to A^+, we obtain the *free monoid* A^* on A.

A word $y \in A^+$ is said to be a *factor* or a *segment* of a word $w \in A^+$ if there exist words $x, z \in A^*$ such that $w = xyz$.

A class of monoids is called a *variety* if it is closed under submonoids, quotient monoids and direct products.

For other semigroup theoretic terminology the reader is referred to [21].

Graphs

A *directed graph* H is a finite non-empty set of objects called *vertices* together with a possibly empty set of ordered pairs of vertices of H called *edges*. A *loop* is an edge that joins a vertex to itself. If more than one edge joins two vertices in the same direction, then H is said to contain *multiple edges*.

By a *graph* G, we mean a directed graph without loops or multiple edges. The set of all vertices (resp., edges) of a graph G is denoted by $V(G)$ (resp., $E(G)$). A graph G is called a *subgraph* of a graph H if $V(G) \subseteq V(H)$ and $(a,b) \in E(G)$ implies $(a,b) \in E(H)$. Then we write $G \subseteq H$. A graph is *finite* if it has a finite set of vertices. A graph G is *transitive*, if $E(G)$ is a transitive binary relation on $V(G)$.

A *complete symmetric graph* is a graph G such that $(u,v) \in E(G)$, for all $u, v \in V(G)$ and $u \neq v$. An infinite countable complete symmetric graph is denoted by K_∞. A *tournament* is a graph G such that, for all distinct $u, v \in G$, either $(u,v) \in E(G)$ or $(v,u) \in E(G)$, but not both.

A graph G is *bipartite* if it is possible to partition $V(G)$ into two subsets V_1 and V_2 such that every element of $E(G)$ joins a vertex in V_1 to V_2 or a vertex in V_2 to V_1.

A *path* in a graph G from x to y is a finite alternating sequence

$$x = x_0, e_1, x_1, e_2, \ldots, e_n, x_n = y$$

of vertices and edges such that $e_i = (x_{i-1}, x_i)$ is an edge of G, for $i = 1, 2, \ldots, n$. A vertex x is said to be an *ancestor* of y if there exists a path from x to y. Two vertices x, y in a graph are *connected* if there exists a finite alternating sequence

$$x = x_0, e_1, x_1, e_2, \ldots, e_n, x_n = y$$

of vertices and edges such that $e_i = (x_{i-1}, x_i)$ or $e_i = (x_i, x_{i-1})$ is an edge of G, for $i = 1, 2, \ldots, n$. The relation 'is connected to' is an equivalence relation. A *connected component* of a graph is a maximal subgraph H such that every pair of vertices, $x, y \in H$ are connected.

A *cycle* in a graph is a path starting and finishing at the same vertex. A graph is said to be *acyclic* if it contains no cycles. It is *null* if it has no edges.

An *infinite ascending chain* (resp., *descending chain*), A_∞ (resp., D_∞), is the graph with the set $\mathbb{Z}^+$ of all positive integers as vertices and with edges (i, j), for all $i < j$ (resp., $i > j$).

For a graph G with edge (x, y), the vertex x is said to be *adjacent to* y and y is said to be *adjacent from* x. The *indegree* of a vertex y is the number of vertices adjacent to y. The *outdegree* of a vertex x is the number of vertices adjacent from x.

For other notation and terminology of graph theory the reader is referred to [11].

Chapter 3

Power D-saturated semigroups

This chapter deals with a combinatorial property defined in terms of power graphs. The *power graph* Pow(S) of a semigroup S has all elements of S as vertices and has edges (u, v), for all $u, v \in S$, $u \neq v$, such that v is a power of u. Figure 3.1 illustrates the power graph of the cyclic group $\mathbb{Z}_8$.

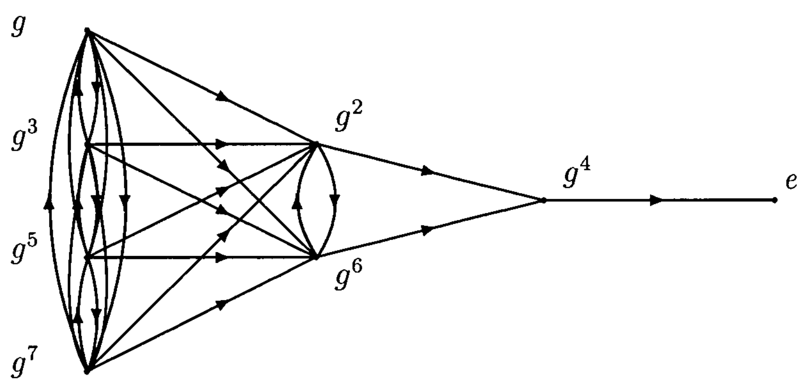


Figure 3.1: The graph whose transitive closure is the power graph of $\mathbb{Z}_8$.

Let D be a finite graph. A semigroup S is said to be *power D-saturated* if and only if, for each infinite subset T of S, the power graph of S has a subgraph isomorphic to D with all vertices being in T (see [31]).

We begin by showing that the class of all power D-saturated semigroups is closed under subsemigroups and homomorphic images, but not direct products.

The first main result describes all groups G and graphs D such that G is D-saturated. This fact underpins the results in the rest of this chapter. Necessary and sufficient conditions are then given for all commutative and linear semigroups possessing the same combinatorial property. After that we describe all semigroups S which are power D-saturated. For each finite abelian group G, we determine the structure of the power graph of G (Theorem 32).

Most of the results in this chapter have been published in [31], [32] and [34].

3.1 Properties of power D-saturated semigroups

Lemma 7 *If D is a graph and S is a power D-saturated semigroup, then all subsemigroups of S are power D-saturated, too.*

Proof. Let U be a subsemigroup of S, and suppose that T is any infinite subset of U. Denote by $\mathrm{Pow}(T)^U$ (resp., $\mathrm{Pow}(T)^S$) the subgraphs induced by the elements of T in U (resp., S).

Evidently, if $(a, b) \in E(\mathrm{Pow}(T)^U)$, then $(a, b) \in E(\mathrm{Pow}(T)^S)$. Suppose that $(a, b) \in E(\mathrm{Pow}(T)^S)$, where $a, b \in U$. The closure of U implies that $b^n \in U$, for all $n \in \mathbb{Z}^+$, and so $(a, b) \in E(\mathrm{Pow}(T)^U)$. Therefore $\mathrm{Pow}(T)^U$ is isomorphic to $\mathrm{Pow}(T)^S$.

Since S is D-saturated, D embeds in $\mathrm{Pow}(T)^S$. Therefore D embeds in $\mathrm{Pow}(T)^U$, and U is D-saturated. □

Lemma 8 *If D is a graph and S is a power D-saturated semigroup, then all quotient semigroups of S are power D-saturated, too.*

Proof. Consider the quotient semigroup S/ρ, where ρ is a congruence. If S/ρ is finite, then there is nothing to prove. Otherwise take any infinite subset T of S/ρ and for each $t_i \in T$ choose a representative $s_i \in S$ such that $\rho s_i = t_i$. Let U be the set of these representatives. Then $\rho^\natural : U \to T$ is a bijection. Clearly, U is infinite, and so D embeds in the subgraph G of $\mathrm{Pow}(S)$ with all vertices of U, since S is D-saturated.

Suppose that $(s_i, s_j) \in E(G)$. Then $s_i = s_j^n$ for some n. Therefore

$$t_i = \rho s_i = \rho s_j^n = (\rho s_j)^n = t_j^n,$$

and so $(t_i, t_j) \in E(\text{Pow}(S/\rho))$. Thus G embeds in $\text{Pow}(S/\rho)$, and so D embeds in $\text{Pow}(S/\rho)$. Therefore S/ρ is D-saturated. $\square$

Example 9 shows that the class of all power D-saturated semigroups is not closed under direct products. Thus power D-saturated semigroups do not form a variety.

Example 9 Let $G = \mathbb{Z}_{p^\infty}$ be a quasicyclic group with generators $g_{i+1}^p = g_i$ and $g_1^p = e_{g_1}$, and let $H = \mathbb{Z}_p$ be a cyclic group with generator g. Both of these groups are power D-saturated for any finite acyclic graph D with edges. However, the direct product $G \times H$ is not D-saturated for any graph D which is not null, since the set of elements $(g_1, g), (g_2, g), (g_3, g), \ldots$ in $G \times H$ induces a null subgraph in $\text{Pow}(G \times H)$.

3.2 Groups

Our first theorem completely describes all pairs (D, G), where D is a finite graph and G is a group such that G is power D-saturated. For the proof we need the following results, the first of which is due to B.H. Neumann:

Lemma 10 ([41]) *A group is centre-by-finite if and only if every infinite sequence contains a pair of elements that commute.*

Lemma 11 (Chinese Remainder Theorem) *Let $m_1, m_2, \ldots, m_r$ be positive integers that are pairwise coprime, and let $a_1, a_2, \ldots, a_r$ be any r integers. Then the system of congruences*

$$x \equiv a_1 (\text{mod } m_1), \ldots, x_r \equiv a_r (\text{mod } m_r)$$

has a solution which is unique modulo $m_1 m_2 \ldots m_r$.

Definition 12 ([49], Definition 3.2) A *logical numbering* on a graph D with n vertices is a function f which assigns to each vertex v of D an integer $f(v)$ such that the following two conditions hold:

(i) each of the integers $1, 2, \ldots, n$ is assigned to exactly one vertex;

(ii) if $(u, v) \in E(D)$, then $f(u) < f(v)$.

Lemma 13 ([49], Theorem 3.8) *A graph has a logical numbering if and only if it is acyclic.*

Lemma 14 *Every finite acyclic graph embeds in A_∞ and D_∞.*

Proof. Let D be a finite acyclic graph with vertex set $v_1, v_2, \ldots v_n$. Then there exists a logical numbering f on D, by Lemma 13. Select any n vertices $u_1, u_2, \ldots u_n$ in D_∞. There exists a unique logical numbering g on the subgraph H induced by the elements $u_1, u_2, \ldots, u_n$, with the property that (u_i, u_j) is an edge in H if and only if $g(u_i) < g(u_j)$, for all $1 \leq i < j \leq n$.

Define $\phi : D \to H$, by $\phi(v_i) = g^{-1}(f(v_i))$. The map ϕ is a bijection, and if $(v_i, v_j) \in E(D)$, then $(\phi(v_i), \phi(v_j)) \in E(H)$. Thus D embeds in H, and hence in D_∞.

The dual argument shows that D embeds in A_∞. □

Definition 15 A graph $D(V, E)$ is called a *direct product* of $D_1(V_1, E_1), \ldots, D_n(V_n, E_n)$ if $V = V_1 \times \cdots \times V_n$ and E is the set of all pairs $((a_1, \ldots, a_n), (b_1, \ldots, b_n))$ such that $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$ and (a_i, b_i) belongs to $E_i \cup \{(v_i, v_i) \mid v_i \in V_i\}$, for all $1 \leq i \leq n$.

Lemma 16 *If $G = \prod_{i=1}^n G_{p_i}$ is a direct product of p_i-groups, where $p_1, \ldots, p_n$ are pairwise distinct primes, then the power graph of G is the direct product of the power graphs of $G_{p_1}, \ldots, G_{p_n}$. That is,*

$$\operatorname{Pow}(G) = \prod_{i=1}^{n} \operatorname{Pow}(G_{p_i}). \tag{3.1}$$

Proof. Suppose that

$$(a_i, b_i) \in E(\operatorname{Pow}(G_{p_i})) \cup \{(v_i, v_i) \mid v_i \in E(G_{p_i})\},$$

for all $i = 1, \ldots, n$. Then $b_i = a_i^{m_i}$ in G_{p_i}, for some positive integer m_i. Denote the order of a_i by k_i. Given that G_{p_i} is a p_i-group, we see that $k_1, \ldots, k_n$ are pairwise coprime. By the Chinese remainder theorem there exists a positive integer k which has remainder m_i upon division by k_i, for all $i = 1, 2, \ldots, n$. If $a = (a_1, a_2, \ldots, a_n) \in G$ and $b = (b_1, b_2, \ldots, b_n) \in G$, then $a^k = b$. Hence $(a, b) \in E(\operatorname{Pow}(G))$.

Obviously, if $(a, b) \in E(\operatorname{Pow}(G))$, then every b_i is a power of a_i, and hence

$$(a_i, b_i) \in E(\operatorname{Pow}(G_{p_i}) \cup \{(v_i, v_i) \mid v_i \in E(G_{p_i})\}).$$

Thus (3.1) follows. □

Lemma 17 *Let p be a prime, and let a and b be two distinct elements in a cyclic p-group $\mathbb{Z}_{p^n}$, where a has order p^r and b has order p^s. Then*

(i) *a belongs to a complete symmetric graph of order $p^r - p^{r-1}$;*

(ii) *$(a, b) \in E(\operatorname{Pow}(\mathbb{Z}_{p^n}))$ if and only if $r \geq s$.*

Proof. (i): Let g be a generator of $\mathbb{Z}_{p^n}$, and denote by $\gcd(a, b)$ the greatest common divisor of two integers a and b. There are precisely $p^r - p^{r-1}$ elements of order p^r in $\mathbb{Z}_{p^n}$, namely all elements g^k such that $\gcd(k, p^n) = p^{n-r}$. Each element of order p^r generates the same subgroup and can be expressed as a power of every other element of the same order. Thus all elements of order p^r induce a complete symmetric graph of order $p^r - p^{r-1}$ in $\operatorname{Pow}(\mathbb{Z}_{p^n})$. When $r = 0$, the identity element forms a complete symmetric graph of order 1.

(ii): Suppose that $r \geq s$. The order of $f = a^{p^{r-s}}$ is equal to p^s. If $a = f$, then $r = s$, and so $|a| = |b|$. All elements of the same order belong to the same complete symmetric graph in $\operatorname{Pow}(\mathbb{Z}_{p^n})$, and hence $(a, b) \in E(\operatorname{Pow}(\mathbb{Z}_{p^n}))$. If $f = b$, then there is nothing to prove, since $(a, f) \in E(\operatorname{Pow}(\mathbb{Z}_{p^n}))$. If $f \neq a, b$, then $|f| = |b|$, and so $(f, b) \in E(\operatorname{Pow}(\mathbb{Z}_{p^n}))$. Then (a, b) is contained in $E(\operatorname{Pow}(\mathbb{Z}_{p^n}))$, since the power relation is transitive.

Conversely, suppose that $(a, b) \in E(\operatorname{Pow}(\mathbb{Z}_{p^n}))$. Take any generator g of $\mathbb{Z}_{p^n}$. Then $a = g^c$, where $\gcd(c, p^n) = p^{n-r}$. Similarly, $b = g^d$, where $\gcd(d, p^n) = p^{n-s}$. Since $a^m = b$, we get $(g^c)^m = g^d$. Thus $mc \equiv d \pmod{p^n}$. This congruence is solvable if and only if $\gcd(c, p^n) | d$. Therefore we see that $\gcd(c, p^n) | \gcd(d, p^n)$, $p^{n-r} | p^{n-s}$ and we get $r \geq s$, as required. □

The next result is also used in Section 3.4.

Lemma 18 *Suppose that G is a torsion group with a finite number of primary components and suppose that each component is finite or quasicyclic. Then every infinite subset T of G contains an infinite subset which induces an infinite chain C in the power graph of G.*

Proof. Let $G = G_{p_1} \times \ldots \times G_{p_n}$, for pairwise distinct prime numbers $p_1, \ldots, p_n$. Recall that $\pi_i : G \to G_{p_i}$ denotes the projection of G onto G_{p_i}, for $1 \le i \le n$.

Take any infinite subset L of G. By induction on $i = 0, 1, \ldots, n$, we define infinite subsets L_i of L such that every image $\pi_k(L_i)$ forms a chain (i.e., a transitive tournament) in the power graph of G_{p_k}, for $k = 1, \ldots, i$. First, put $L_0 = L$. Suppose that the set L_i has already been defined for some $0 \le i < n$.

If $\pi_{i+1}(L_i)$ is finite, then we can find an infinite subset L_{i+1} of L_i such that $\pi_{i+1}(L_{i+1})$ has only one element, and so forms a chain. (Note that in this part of the proof consecutive repetitions of the same vertex in a chain are allowed, i.e., loops are attached to all vertices of the graphs.)

Next, consider the case where $\pi_{i+1}(L_i)$ is infinite. Then $G_{p_{i+1}}$ is infinite too, and so it is quasicyclic. Putting $p = p_{i+1}$, we get $G_p = \mathbb{Z}_{p^\infty}$. Since $|\pi_{i+1}(L_i)| = \infty$ and $\mathbb{Z}_{p^\infty}$ is the union of an ascending chain of cyclic groups, we can choose an infinite sequence $t_1, t_2, \ldots \in L_i$ such that each element $\pi_{i+1}(t_j)$ has order p^{ℓ_j}, for $j = 1, 2, \ldots$ and $\ell_1 < \ell_2 < \ldots$. Take any positive integers $j < k$. There exists a cyclic subgroup $\mathbb{Z}_{p^\ell}$ of $\mathbb{Z}_{p^\infty}$ such that both $\pi_{i+1}(t_j)$ and $\pi_{i+1}(t_k)$ belong to $\mathbb{Z}_{p^\ell}$. Lemma 17 shows that $\pi_{i+1}(t_j)$ is a power of $\pi_{i+1}(t_k)$. It follows that the sequence $\pi_{i+1}(t_1), \pi_{i+1}(t_2), \ldots$ forms an infinite chain in the power graph of G. We can take $L_{i+1} = \{t_1, t_2, \ldots\}$.

Thus we have defined the sets $L_1, \ldots, L_n$. All projections of the infinite set L_n form ascending chains in $G_{p_1}, \ldots, G_{p_n}$. Lemma 16 implies that L_n induces an infinite chain C in the power graph of G. □

Clearly, for each group G and every set of vertices V, there exist maximal graphs $D(V, E)$ such that G is D-saturated. Theorem 19 shows that in fact there are only three types of maximal graphs: null graphs, transitive subtournaments of D_∞, and complete graphs.

Theorem 19 *Let $D(V, E)$ be a finite graph with $E \neq \emptyset$, and let G be an infinite group. Then G is power D-saturated if and only if G is a centre-by-finite* *torsion group, the centre $C(G)$ has a finite number of primary components, each primary component of $C(G)$ is finite or quasicyclic, the order of $G/C(G)$ is not divisible by p for each quasicyclic p-subgroup of G, and D is isomorphic to a subgraph of D_∞.*

Proof. The 'only if' part. Suppose that G is power D-saturated, i.e., every infinite subset of G contains a power subgraph isomorphic to D. Hence every infinite subset has at least two elements a, b such that b is a power of a, and so a and b commute. Lemma 10 implies that G is centre-by-finite.

If G has an element g of infinite order, then the vertices $g^2, g^3, g^5, \ldots$ are not adjacent in the power graph of G. Since $E(D)$ contains edges between distinct vertices and G is D-saturated, we see that G has to be torsion.

If G contains elements g_i of orders p_i, for infinitely many primes $p_1, p_2, \ldots$, then the vertices $g_1, g_2, \ldots$ are not adjacent in $\mathrm{Pow}(G)$. This contradicts the D-saturation of G again. Therefore G has a finite number of primary components.

If a p-primary component $C(G)_p$ of the centre $C(G)$ has infinite p-rank, then $C(G)_p$ contains an infinite independent subset $\{g_1, g_2, \ldots\}$ (see [50], 4.2). Clearly, these elements are not adjacent in the power graph of $C(G)$. Thus the p-rank of $C(G)_p$ is finite.

It follows that $C(G)_p$ is a direct product of finitely many cyclic or quasicyclic groups (see [50], 4.3.13). Suppose that $C(G)_p$ is infinite, but is not quasicyclic. Then it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^\infty}$, and there exists an infinite set of elements which induces a null subgraph in the $\mathrm{Pow}(G)$, as seen in Example 9. Therefore G is not D-saturated. Thus each primary component of $C(G)$ is finite or quasicyclic.

Take any prime number p such that G has a quasicyclic subgroup $\mathbb{Z}_{p^\infty}$. If $g_1, g_2, \ldots$ are the same generators of $\mathbb{Z}_{p^\infty}$ in Example 9, then we see that they induce a subgraph of the power graph of G isomorphic to D_∞, that is $(g_i, g_j) \in E(\mathrm{Pow}(G))$ if and only if $i > j$. Since G is D-saturated, D is a subgraph of D_∞.

Suppose that p divides $|G/C(G)|$ and that G has a quasicyclic subgroup $\mathbb{Z}_{p^\infty}$ with generators $g_1, g_2, \ldots$ in Example 9. Pick an element h in G such that its image $hC(G)$ has order p in $G/C(G)$. Then all vertices $(h, g_1), (h, g_2), \ldots$ are not adjacent in $\mathrm{Pow}(G)$, and so G is not D-saturated. This contradiction shows that $|G/C(G)|$ is not divisible by p for each quasicyclic p-subgroup of G.

The 'if' part. Assume that D has edges, G is a torsion group with a finite number of primary components, each primary component of G is finite or quasicyclic, and the order of $G/C(G)$ is not divisible by p for each quasicyclic p-subgroup of G. Take any infinite set T of S. Then by Lemma 18, there exists an infinite subset L of T which induces an infinite chain C in the power graph of G.

Induction on the number of ancestors in C shows that every vertex of C is finite. Hence C is isomorphic to D_∞. Thus D embeds in D_∞, by Lemma 14, which completes our proof. □

3.3 Nil semigroups

The following infinite version of Ramsey's Theorem ([47], Theorem A) is needed.

Lemma 20 (Ramsey Theorem) *Let S be an infinite set. For all positive integers k, r, if we colour the k-element subsets of S in r colours, then there always exists an infinite subset T of S such that all the k-element sets of T have the same colour.*

This lemma easily gives us the following

Corollary 21 *Every infinite graph contains an infinite ascending chain, an infinite descending chain, an infinite complete symmetric graph, or an infinite set of vertices that induce a null subgraph.*

Proof. Let $D = (V, E)$ be an infinite graph. Choosing a countable subset of V and indexing its elements by the positive integers, we may assume that $V = \{v_1, v_2, \ldots\}$. Let us assign a colour $c(\{v_i, v_j\})$ to each set $\{v_i, v_j\}$, where $1 \leq i < j$ by the rule

$$c(\{v_i, v_j\}) = \begin{cases} \text{red} & \text{if } (v_i, v_j), (v_j, v_i) \notin E, \\ \text{yellow} & \text{if } (v_i, v_j) \in E, (v_j, v_i) \notin E, \\ \text{green} & \text{if } (v_j, v_i) \in E, (v_i, v_j) \notin E, \\ \text{blue} & \text{if } (v_i, v_j), (v_j, v_i) \in E. \end{cases}$$

Lemma 20 tells us that there exists an infinite subset W of V such that all two-element subsets of W have the same colour c. Denote by G the subgraph of D induced by W. If c is red, then G is a null graph. If c is yellow, then G is an infinite ascending chain. If c is green, then G is isomorphic to an infinite descending chain. If c is blue then G is an infinite complete symmetric graph. $\square$

Lemma 22 *If S is an infinite nil semigroup, then S has an infinite subset that induces a null subgraph in the power graph of S.*

Proof. Let 0 be the zero of S. Suppose to the contrary that $\mathrm{Pow}(S)$ has no infinite subsets which induce null subgraphs. Evidently, all elements of every infinite path $x_1, x_2, \ldots$ of a power graph of a semigroup with edges (x_i, x_j), for all $1 \leq i < j$ are not nil. Therefore $\mathrm{Pow}(S)$ has an infinite path, $x_1, x_2, \ldots$ with edges (x_j, x_i), for all $1 \leq i < j$, by Corollary 21. We may assume that $x_1 \neq 0$, since otherwise we can start the path with x_2.

Consider two elements x_2x_i and x_2x_j in S, where $2 < i < j$. Since x_1, x_2, x_i and x_j are vertices of our path, we know that there exist positive integers $k_1, k_2, k_3, k_4 > 1$ such that

$$x_2^{k_1} = x_1, \tag{3.2}$$
$$x_i^{k_2} = x_j^{k_3} = x_2, \tag{3.3}$$
$$x_j^{k_4} = x_i. \tag{3.4}$$

Then $x_1 = x_2x_i(x_i^{k_2-1}x_2^{k_1-2})$ and $x_2x_j(x_j^{k_4-1}) = x_2x_i$, and so x_2x_i and x_2x_j are nonzero elements in S.

Suppose that x_2x_i and x_2x_j are adjacent in the power graph $\mathrm{Pow}(S)$. If

$$(x_2x_j)^k = x_2x_i,$$

for some positive integer k, then by combining equations (3.3) and (3.4), and equating indices we get

$$k(k_3 + 1) = k_3 + k_4.$$

If $k = 1$, then $k_4 = 1$, a contradiction. If $k > 1$, then

$$k(k_3 + 1) > 2(k_3) > k_3 + k_4 = k(k_3 + 1),$$

and again we have a contradiction.

On the other hand, if $(x_2x_i)^k = x_2x_j$, for some $k > 1$, we get

$$k(k_3 + k_4) = k_3 + 1.$$

Then

$$k(k_3 + k_4) > k_3 + k_4 > k_3 + 1 = k(k_3 + k_4),$$

and another contradiction arises.

These contradictions show that all elements $x_2x_3, x_2x_4, x_2x_5, \ldots$ are not adjacent in $\mathrm{Pow}(S)$, which completes the proof. □

3.4 Commutative semigroups

The following theorem describes all power D-saturated commutative semigroups.

Theorem 23 *Let $D = (V, E)$ be a finite graph with $E \neq \emptyset$, and let S be an infinite commutative semigroup. Then S is power D-saturated if and only if the following conditions hold:*

(i) *D is acyclic;*

(ii) *S is periodic;*

(iii) *S has a finite number of idempotents;*

(iv) *all but a finite number of elements of S belong to the union of all subgroups of S;*

(v) *every subgroup of S has a finite number of primary components;*

(vi) *for every prime p, each p-subgroup of S is either finite or quasicyclic.*

Proof. Let S be a semilattice Y of Archimedean semigroups S_y.

The 'only if' part. Suppose that S is power D-saturated.

If S has an element s which is not periodic, then the vertices $s^2, s^3, s^5, \ldots$ are not adjacent in the power graph of S. Then S is not power D-saturated, since D is not null. Hence S is periodic, i.e., (ii) holds.

By Lemma 6, every periodic Archimedean semigroup S_y has a unique idempotent e_y and $G_y = e_y S_y$ is a group. If S contains infinitely many Archimedean components, then the idempotents of these components are not adjacent in $\mathrm{Pow}(S)$. This contradiction shows that Y is finite, and that (iii) holds.

Suppose to the contrary that (iv) is not satisfied. Then the nilextension $N_y = S_y/G_y$ is infinite, for some $y \in Y$. Consider two cases.

Case 1: All paths in $\mathrm{Pow}(N_y)$ are finite. Then the elements of zero indegree generate N_y. Since every element in N_y has finite order, $\mathrm{Pow}(N_y)$ contains infinitely many elements of zero indegree. Of course, all these vertices are not adjacent in $\mathrm{Pow}(N_y)$.

Case 2: $\mathrm{Pow}(N_y)$ contains an infinite path $\{n_1, n_2, \ldots\}$. By Lemma 22, N_y contains an infinite subset which induces a null subgraph in $\mathrm{Pow}(N_y)$.

In both cases N_y contains and infinite null subgraph, and is not D-saturated. Lemma 8 implies that S_y is not D-saturated. Therefore S in not D-saturated, by Lemma 7, a contradiction. Thus the nilsemigroup N_y is finite. Since Y is finite, we see that (iv) holds.

Suppose that there exists $y \in Y$ such that the group G_y contains elements $g_1, g_2, \ldots$ of prime orders $p_1, p_2, \ldots$, for infinitely many distinct primes. Then the vertices $g_1, g_2, \ldots$ are not adjacent in $\mathrm{Pow}(G_y)$, and so $\mathrm{Pow}(S)$ is not D-saturated, by Lemma 7. This contradiction shows that G_y has a finite number of primary components, for all $y \in Y$, i.e., (v) holds.

The facts that all primary p-components of G_p are finite or quasicyclic, and that every p-subgroup of S is either finite or quasicyclic follow using the same reasoning as in Theorem 19. Thus (vi) holds.

Take any Archimedean component which contains a quasicyclic subgroup $\mathbb{Z}_{p^\infty}$. Since S is infinite, there exists at least one such subgroup. If $g_1, g_2, \ldots$ are the same generators of $\mathbb{Z}_{p^\infty}$ in Example 9, then we see that they induce a subgraph of the power graph of S isomorphic to D_∞, that is (g_i, g_j) is an edge in $\mathrm{Pow}(S)$ if and only if $i > j$. Since S is D-saturated, D is a subgraph of D_∞. Therefore D embeds in this subgraph, and so it is acyclic, i.e., (i) holds.

The 'if' part. Consider an infinite subset T of S. Since S contains a finite number of Archimedean components S_y and each nil semigroup S_y/G_y is finite, there exists $y \in Y$ and an infinite subset $U \subseteq T$ such that all elements of U belong to the group G_y. Therefore it suffices to verify that G_y is power D-saturated. By (v),

$$G_y = G_{p_1} \times \ldots \times G_{p_n},$$

for pairwise distinct primes $p_1, \ldots, p_n$, and by (vi), G_{p_i} is either finite or quasicyclic. Therefore Lemma 18 tells us that there exists an infinite subset L of T which induces an infinite chain C in the power graph of G_y.

Thus all projections of the infinite set L form chains in $G_{p_1}, \ldots, G_{p_n}$. Then by Lemma 16, L induces an infinite chain C in the power graph of G_y. Thus D embeds in C, by Lemma 14, and so G_y is power D-saturated. Hence S is power D-saturated. □

3.5 Linear semigroups

Put $M_j = \{a \in M_n(K) \mid \text{rank}(a) \leq j\}$. The following technical lemmas, found in [23] and [45] (see also [27]), are used in this section.

Lemma 24 ([23], Lemma 3) *Let G be a group. Then the monomial matrix semigroup $M_n(G)$ is an inverse semigroup with the only ideals*

$$\{0\} = M_0 \subset M_1 \subset \ldots \subset M_n = M_n(G),$$

where $M_j = \{s \mid s$ has at most j nonzero entries $\}$. Moreover,

$$M_j/M_{j-1} \cong \mathcal{M}\left(G_j, \binom{n}{j}, \binom{n}{j}, \Delta\right),$$

where G_j is an extension of $G^j = G \times \ldots \times G$ by the symmetric group S_j and Δ is the identity matrix. All idempotents of $M_n(G)$ are diagonal and a power of every element is diagonal.

Lemma 25 ([45], Theorem 2.3) *The sets*

$$\{0\} = M_0 \subset M_1 \subset \ldots \subset M_n = M_n(K)$$

are the only ideals of the monoid $M_n(K)$. Each Rees factor M_j/M_{j-1} is isomorphic to the completely 0-simple semigroup $\mathcal{M}(GL_j(K), X_j, Y_j, Q_j)$, where the matrix $Q_j = (q_{yx})$ is defined for $x \in X_j, y \in Y_j$, by $q_{yx} = yx$ if yx is of rank j and 0 otherwise.

In all results related to linear and monomial semigroups, the symbol M_n is used to mean one of $M_n(K)$ or $M_n(G)$.

Theorem 26 *Let $D = (V, E)$ be a finite graph with $E \neq \emptyset$, K a skew field, G a group, and let S be an infinite matrix semigroup in $M_n(K)$ or in $M_n(G)$. Then S is power D-saturated if and only if D is acyclic and all but a finite number of elements of S are contained in the union of a finite number of centre-by-finite torsion groups H_i, where $i = 1, \ldots, k$, such that the centre $C(H_i)$ of each H_i has a finite number of primary components, each primary component of $C(H_i)$ is finite or quasicyclic, and the order of $H_i/C(H_i)$ is not divisible by p for each quasicyclic p-subgroup of H_i.*

Proof. In the full linear or monomial semigroup M_n, let us consider the chain of ideals $M_0, M_1, \ldots, M_n$, defined in Lemma 25 and Lemma 24.

The 'only if' part. Suppose that S is power D-saturated. Then for every infinite subset of S the graph $D \subseteq \text{Pow}(S)$.

Consider the set T of all elements of S contained in the $\mathcal{H}$-classes of M_n which are not groups. Suppose that T is infinite. By the definition of T and M_j, it follows that

$$T \subseteq S \subseteq M_n = \cup\{M_j \setminus M_{j-1} \mid 1 \leq j \leq n\}.$$

Therefore $T = \cup\{T \cap (M_j \setminus M_{j-1} \mid 1 \leq j \leq n\}$, and hence at least one $T \cap (M_j \setminus M_{j-1})$ must be infinite, for some $1 \leq j \leq n$. If $s \in T \cap (M_j \setminus M_{j-1})$, then [21], Lemma 3.2.7 shows that $s^k \in M_{j-1}$, for all $k \geq 2$. Hence the elements in $T \cap (M_j \setminus M_{j-1})$ induce an infinite null subgraph in $\text{Pow}(S)$. Since D has edges, we see that D does not embed in this subgraph. This contradicts the power D-saturation of S and shows that T is finite.

Suppose that the elements of $S \setminus T$ belong to infinitely many $\mathcal{H}$-classes of M_n. The definition of T shows that all these $\mathcal{H}$-classes are groups. By the axiom of choice (see [38], Appendix A), we can form a subset Q that contains exactly one element of each $\mathcal{H}$-class of M_n intersecting $S \setminus T$. Then Q is infinite and induces an infinite null subgraph in $\mathrm{Pow}(S)$. Again, this contradicts power D-saturation, and shows that $S \setminus T$ is contained in a finite number of $\mathcal{H}$-classes of M_n.

Take any $\mathcal{H}$-class $G^j_{i\lambda}$ of M_n intersecting $S \setminus T$. Put $R = G^j_{i\lambda} \cap S$. If R has an element r of infinite period, then the vertices $r^2, r^3, r^5, r^7, \ldots$ are not adjacent in the power graph of S. This contradicts power D-saturation again, and shows that all elements of R are periodic. Since $G^j_{i\lambda}$ is a group and R is periodic, we see that R is a subgroup of $G^j_{i\lambda}$, by Lemma 1.

Thus $S \setminus T$ is a union of a finite number of groups. By Lemma 7, the power D-saturation is inherited by subsemigroups, and so all these groups are also power D-saturated. Evidently, all but a finite number of elements of $S \setminus T$ are contained in the union of a finite number of infinite groups H_i, where $i = 1, \ldots, k$. By Theorem 19, each H_i is a centre-by-finite torsion group such that the centre $C(H_i)$ of H_i has a finite number of primary components, each primary component of $C(H_i)$ is finite or quasicyclic and the order of $H_i/C(H_i)$ is not divisible by p for each quasicyclic p-subgroup of H_i.

It remains to verify that the graph D is acyclic. Since S is infinite, it has an infinite subgroup R, which contains a quasicyclic subgroup $\mathbb{Z}_{p^\infty}$. The generators of $\mathbb{Z}_{p^\infty}$ in Example 9 induce a subgraph which is isomorphic to D_∞, as seen in Theorem 19. By the power D-saturation of S, we see that D embeds in the chain, and so D is acyclic too.

The 'if' part. By Theorem 19, we may assume that all but a finite number of elements of S are contained in the union of a finite number of power D-saturated groups. Then every infinite subset T of S contains an infinite subset $U = S \cap R$, where R is a group. By assumption, R is power D-saturated, and therefore D embeds in the subgraph of $\mathrm{Pow}(R)$ induced by the vertices of U. Hence D embeds in $\mathrm{Pow}(S)$. □

3.6 Arbitrary semigroups

Before proceeding it is worth mentioning that the power graph of a D-saturated semigroup S must have a finite number of connected components, since otherwise the subgraph induced by selecting one element from each component is null. Moreover, it is not difficult to show that each connected component contains exactly one idempotent. However, it is not true that each connected component forms a subsemigroup in S. For example, Figure 3.2 illustrates the power graph of the Brandt semigroup

$$B_2 = \{a, b \mid aba = a, bab = b, a^2 = b^2 = 0\}.$$

It has edges $(a, 0), (b, 0)$, but $\{a, b, 0\}$ is not a subsemigroup of B_2.

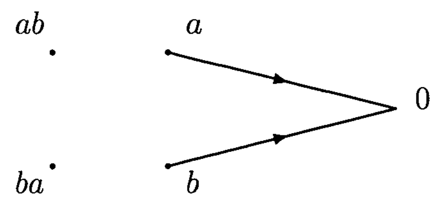


Figure 3.2: The power graph of the Brandt semigroup B_2.

The following result is used in the next theorem.

Lemma 27 ([52], Proposition 11.1) *An epigroup with finitely many idempotents has a finite ideal series in which each factor is either completely simple (if it is the kernel) or completely 0-simple or a nilsemigroup and, in the first two cases, the Rees matrix semigroups have finite sandwich matrices.*

The next theorem describes all nontrivial pairs (D, S) such that D is a finite graph and S is power D-saturated.

Remark 28 It is appropriate to note that a description of all pairs D and S could also be obtained using the concepts arising in [52], Proposition 11.2 and [52], Proposition 12.2. That is, it can be shown that an infinite semigroup S is D-saturated if and only if it is *finitely assembled* (see [52] §11) from a finite number of D-saturated semigroups.

For the convenience of the reader a self-contained proof follows.

Theorem 29 *Let D be a finite graph that is not null, and let S be an infinite semigroup. Then the following conditions are equivalent:*

(i) *the power graph of S is D-saturated;*

(ii) *D is acyclic and S^0 has a finite ideal series*

$$\emptyset = S_0 \subset S_1 \subset \ldots \subset S_n = S^0,$$

where every infinite S_i/S_{i-1} is a Rees matrix semigroup that has a finite sandwich matrix with entries in a centre-by-finite torsion group H_i such that each primary component of the centre of H_i is finite or quasicyclic, the centre of H_i has a finite number of primary components, and the index of the centre is not divisible by p for each quasicyclic p-subgroup of H_i.

Proof. (i)$\Rightarrow$(ii): If S has an element s that is not periodic, then the vertices $s^2, s^3, s^5, \ldots$ are not adjacent in $\mathrm{Pow}(S)$. Obviously, if an infinite sequence of vertices induces a null subgraph of $\mathrm{Pow}(S)$, then S is not power D-saturated. This contradiction shows that S is periodic, and so it is an epigroup.

If S contains infinitely many idempotents, then the idempotents are not adjacent in $\mathrm{Pow}(S)$, a contradiction. Therefore S contains a finite number of idempotents.

By Lemma 27, S has a finite ideal series

$$\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S,$$

where each factor S_i/S_{i-1} is nil, completely simple or completely 0-simple with finite sandwich matrix. Hence S^0 has a finite ideal series, where each factor S_i/S_{i-1} is nil or completely 0-simple with finite sandwich matrix.

Consider an infinite factor S_i/S_{i-1}. (Since S is infinite, there exists at least one factor like this.) If S_i/S_{i-1} is nil, then Lemma 22 tells us that S_i/S_{i-1} contains an infinite subset that induces a null subgraph in $\mathrm{Pow}(S_i/S_{i-1})$. Hence S_i has a quotient semigroup which is not D-saturated, and so S_i is not D-saturated, by Lemma 8. Then S is not D-saturated, by Lemma 7. Thus we see that every infinite factor of the ideal series is completely 0-simple.

Suppose that the sandwich matrix of $S_i/S_{i-1} = M^0(H; I, \Lambda; P)$ has a zero entry $p_{\lambda k}$, for some $k \in I, \lambda \in \Lambda$. Then $G_{k\lambda}^2 = 0$, by Lemma 5(vii). Hence $s^\ell \in S_{i-1}$, for all $s \in G_{k\lambda}$ and $\ell > 1$. This means that the elements of $G_{k\lambda}$ induce a null subgraph in $\mathrm{Pow}(S_i/S_{i-1})$. We know by Lemma 5(iv) that all $\mathcal{H}$-classes in S_i/S_{i-1} have the same cardinality. Therefore $G_{k\lambda}$ is infinite, since I and Λ are finite. Then $\mathrm{Pow}(S_i/S_{i-1})$ contains an infinite null subgraph, and S_i/S_{i-1} is not D-saturated. It follows that S is not D-saturated, by Lemma 8 and Lemma 7. This contradiction shows that all entries of P are nonzero. Moreover, by Lemma 5(vi), these subgroups are isomorphic.

All subsemigroups of S are power D-saturated, by Lemma 7. Thus we know by Theorem 19 that, if S_i/S_{i-1} is infinite, then every $\mathcal{H}$-class of S_i/S_{i-1} is an isomorphic copy of a centre-by-finite torsion group H, such that the centre $C(H)$ has a finite number of primary components, each primary component of $C(H)$ is finite or quasicyclic and the order of $H/C(H)$ is not divisible by p for each quasicyclic p-subgroup of H.

Take any infinite $\mathcal{H}$-class. It contains a quasicyclic subgroup $\mathbb{Z}_{p^\infty}$. Then the generators of $\mathbb{Z}_{p^\infty}$ (see Example 9) induce a subgraph that is isomorphic to D_∞, which is of course acyclic. By the power D-saturation of S, D embeds in this subgraph, and so D is acyclic too.

(ii)$\Rightarrow$(i): Pick any infinite subset T of S. Then

$$T = \cup\{T \cap (S_i \setminus S_{i-1} \mid 1 \le i \le n\},$$

and so at least one $T \cap (S_i \setminus S_{i-1})$ is infinite, for some $1 \le i \le n$. Since all entries of the sandwich matrix of the Rees matrix semigroup $S_i \setminus S_{i-1}$ are nonzero, Lemma 5(vi) implies that $S_i \setminus S_{i-1}$ is the union of 0 and a finite number of subgroups isomorphic to H. All of these subgroups are power D-saturated, by Theorem 19. Clearly, at least one of these subgroups has an infinite intersection X with T. It follows that D embeds in the subgraph induced by the vertices of $X \subseteq T$, and hence S is power D-saturated. □

3.7 The structure of Pow(S) for abelian groups

In order to describe the power graphs of all finite abelian groups, the following result is needed.

Lemma 30 ([50], 4.2.6) *An abelian group G is finite if and only if it is the direct product of finitely many cyclic groups with prime power orders.*

Then for any finite abelian group G and any elements a, b in G, the following notation may be used.

Denote the primary components of G by $G_{p_1}, \ldots, G_{p_n}$, and express each G_{p_i} as a direct product of cyclic groups

$$G_{p_i} = (\mathbb{Z}_{p_i^{w_{i,1}}})^{q_{i,1}} \times (\mathbb{Z}_{p_i^{w_{i,2}}})^{q_{i,2}} \times \ldots \times (\mathbb{Z}_{p_i^{w_{i,m_i}}})^{q_{i,m_i}},$$

where $w_{i,1} < w_{i,2} < \ldots < w_{i,m_i}$. For $i = 1, \ldots, n$, denote the projections of a and b on G_{p_i} by a_i and b_i, respectively. Choose generators $g_{i,j,k}$ in the cyclic groups of G_{p_i} above, where $1 \leq j \leq m_i$ and $1 \leq k \leq q_{i,j}$. Write a_i and b_i in the form $a_i = g_{i,1,1}^{c_{i,1,1}} \ldots g_{i,m_i,q_{i,m_i}}^{c_{i,m_i,q_{i,m_i}}}$, and $b_i = g_{i,1,1}^{d_{i,1,1}} \ldots g_{i,m_i,q_{i,m_i}}^{d_{i,m_i,q_{i,m_i}}}$, where $c_{i,j,k} = u_{i,j,k}{p_i}^{w_{i,j}-r_{i,j,k}}$, $d_{i,j,k} = v_{i,j,k}{p_i}^{w_{i,j}-s_{i,j,k}}$ and $\gcd(u_{i,j,k}, p_i) = 1$, $\gcd(v_{i,j,k}, p_i) = 1$.

The following definition is also needed.

Definition 31 ([48], Definition 2.6) Let m be a positive integer. The *Euler phi function* is the cardinality of the set $\{1 < n < m : \gcd(n, m) = 1\}$ and is denoted by $\phi(m)$.

Theorem 32 *Let G be a finite abelian group, and let a, b be any elements of G. Suppose that the prime factorisation of the order of a is $|a| = \prod_{i=1}^{n} {p_i}^{t_i}$, where $1 \leq t_i \leq w_{i,m_i}$. Then*

(i) *a belongs to a complete symmetric graph of order*

$$\prod_{i=1}^{n} ({p_i}^{t_i} - {p_i}^{t_i-1}),$$

where we replace $({p_i}^{t_i} - {p_i}^{t_i-1})$ by 1 if $t_i = 0$;

(ii) *(a, b) is an edge of* $\mathrm{Pow}(G)$ *if and only if, for every $i = 1, \ldots, n$,*

$${p_i}^{w_{i,j}} | v_{i,j,k} u_{i,j,k}^{\phi({p_i}^{w_{i,j}})-1} {p_i}^{r_{i,j,k}-s_{i,j,k}} - v_{i,j',k'} u_{i,j',k'}^{\phi({p_i}^{w_{i,j'}})-1} {p_i}^{r_{i,j',k'}-s_{i,j',k'}},$$

for all $1 \leq j \leq j' \leq m_i$, and $1 \leq k \leq k' \leq q_{i,j'}$;

(iii) *If w_{i,h_i} is the smallest exponent in G_{p_i} such that $t_i \leq w_{i,h_i}$, then the power graph of G contains*

$$\prod_{i=1}^{n} \frac{(p_i^{w_{i,1}})^{q_{i,1}} \ldots (p_i^{w_{i,h_{i-1}}})^{q_{i,h_{i-1}}} ((p_i^{t_i})^{q_{i,h_i}+\ldots+q_{i,m_i}} - (p_i^{t_i-1})^{q_{i,h_i}+\ldots+q_{i,m_i}})}{(p_i^{t_i} - p_i^{t_i-1})}$$

complete symmetric graphs of order $\prod_{i=1}^{w} ({p_i}^{t_i} - {p_i}^{t_i-1})$, for each t_i. If $t_i = 0$ for any i, then we replace $({p_i}^{t_i} - {p_i}^{t_i-1})$ by 1.

Proof. It is enough to focus on a primary component of G, verify all formulas, and then apply Lemma 16 to obtain complete results. To simplify notation we drop all references involving i throughout the proof. In other words, we fix i and put $p = p_i$, $t = t_i$, $w_s = w_{i,s}$, etc.

Each element of order p^t in G_p belongs to a complete symmetric graph of order $p^t - p^{t-1}$. Since the orders of elements in different p-components are mutually coprime, the formula in (i) follows from Lemma 16.

Consider the primary p-component

$$G_p = (\mathbb{Z}_{p^{w_1}})^{q_1} \times (\mathbb{Z}_{p^{w_2}})^{q_2} \times \ldots \times (\mathbb{Z}_{p^{w_m}})^{q_m},$$

where the k^{th} copy of $\mathbb{Z}_{p^{w_j}}$ has generator $g_{j,k}$, for $1 \leq j \leq m$ and $1 \leq k \leq q_j$. Assume $w_1 < w_2 < \ldots < w_m$. Let $a = g_{1,1}^{c_{1,1}} \ldots g_{m,q_m}^{c_{m,q_m}}$ and $b = g_{1,1}^{d_{1,1}} \ldots g_{m,q_m}^{d_{m,q_m}}$ be two elements of G_p, where $g_{j,k}^{c_{j,k}}$ and $g_{j,k}^{d_{j,k}}$ have orders $p^{r_{j,k}}$ and $p^{s_{j,k}}$, respectively. Solving $a^x = b$ yields the system of congruences:

$$xc_{j,k} \equiv d_{j,k} \bmod (p^{w_j}), \text{ for all } j, k. \tag{3.5}$$

Each congruence, when considered in isolation, is solvable if and only if $\gcd(c_{j,k}, p^{w_j})$ divides $d_{j,k}$. This implies that $r_{j,k} \geq s_{j,k}$, by Lemma 17. Moreover, since $g_{j,k}^{c_{j,k}}$ has order $p^{r_{j,k}}$, we see that $c_{j,k}$ can be expressed as $u_{j,k}p^{w_j - r_{j,k}}$, where $\gcd(u_{j,k}, p) = 1$. Similarly, $d_{j,k} = v_{j,k}p^{w_j - s_{j,k}}$, where $\gcd(v_{j,k}, p) = 1$, for a positive integer $v_{j,k}$. Thus (3.5) gives us

$$\begin{aligned} xu_{j,k} &\equiv v_{j,k}p^{r_{j,k}-s_{j,k}} \bmod (p^{w_j}) \\ x &\equiv v_{j,k}u_{j,k}^{\phi(p^{w_j})-1}p^{r_{j,k}-s_{j,k}} \bmod (p^{w_j}), \end{aligned} \tag{3.6}$$

where ϕ is the Euler phi-function.

Thus $(a, b) \in E(\operatorname{Pow}(G_p))$ if and only if there exists a solution to (3.6). The system of congruences in (3.6) is solvable if and only if

$$p^{w_j} \quad | \quad v_{j,k}u_{j,k}^{\phi(p^{w_j})-1}p^{r_{j,k}-s_{j,k}} - v_{j',k'}u_{j',k'}^{\phi(p^{w_{j'}})-1}p^{r_{j',k'}-s_{j',k'}},$$

for $1 \leq j \leq j' \leq m$ and $1 \leq k \leq k' \leq q_{j'}$. The formula in (ii) follows directly from Lemma 16.

We observe that $|a| \geq |b|$ is a necessary, but not sufficient condition for $(a, b) \in E(\operatorname{Pow}(G_p))$. In $\mathbb{Z}_4 \times \mathbb{Z}_4 = \langle a \rangle \times \langle b \rangle$ we have $|ab| = 4$ and $|a^2| = 2$, but (ab, a^2) does not belong to $E(\operatorname{Pow}(\mathbb{Z}_4 \times \mathbb{Z}_4))$. Moreover,

$\mathrm{Pow}(G_p)$ is not the direct product of the power graphs of its components as in Lemma 16, for $(a, a^2) \in E(\mathrm{Pow}(\mathbb{Z}_4))$ and $(b, b^3) \in E(\mathrm{Pow}(\mathbb{Z}_4))$, but $(ab, a^2b^3) \notin E(\mathrm{Pow}(\mathbb{Z}_4 \times \mathbb{Z}_4))$.

Next, we count the number of complete symmetric graphs in $\mathrm{Pow}(G)$. Suppose that $|a| = p^t$ in G_p. At least one $g_{j,k}$ has order p^t in $(\mathbb{Z}_{p^{w_j}})^{q_k}$, where $w_j \geq t$. Choose h such that w_h is the minimum integer with this property.

For $j < h$, rewrite G_p in the form

$$(\mathbb{Z}_{p^1})^{f_1}(\mathbb{Z}_{p^2})^{f_2} \dots (\mathbb{Z}_{p^{h-1}})^{f_{h-1}},\text{, where } f_j \geq 0. \tag{3.7}$$

The number of elements of order p^t is obtained by summing over all possible combinations of elements of order up to p^t in G_p. For ease in notation let $y = q_h + \dots + q_m$ and $e_i = f_{h-1} + \dots + f_{h-i}$, for $i = 1, \dots h-1$. Then the number of elements of order p^t is

$$\sum_{z_1=1}^{y} \binom{y}{z_1} (p^t - p^{t-1})^{z_1} . \Bigg(\sum_{z_2=0}^{e_1+y-z_1} \binom{e_1 + y - z_1}{z_2} (p^{t-1} - p^{t-2})^{z_2}$$
$$\cdot \Bigg(\sum \cdots \sum_{z_h=0}^{e_{h-1}+y-(z_1+\dots+z_{h-1})} \binom{e_{h-1} + y - (z_1 + \dots + z_{h-1})}{z_h} (p-1)^{z_h} \Bigg) \dots \Bigg) \Bigg). \tag{3.8}$$

Now (3.8) is a series of nested binomial expansions and simplifies to

$$(p^1)^{f_1}(p^2)^{f_2} \dots (p^{h-1})^{f_{h-1}}((p^t)^{q_h+\dots+q_m} - (p^{t-1})^{q_h+\dots+q_m}). \tag{3.9}$$

Resubstituting, (3.9) becomes

$$(p^{w_1})^{q_1}(p^{w_2})^{q_2} \dots (p^{w_{h-1}})^{q_{h-1}}((p^t)^{q_h+\dots+q_m} - (p^{t-1})^{q_h+\dots+q_m}).$$

Each element of order p^t in G_p belongs to a complete symmetric graph of order $(p^t - p^{t-1})$. Therefore $\mathrm{Pow}(G_p)$ has exactly

$$\frac{(p^{w_1})^{q_1}(p^{w_2})^{q_2} \dots (p^{w_{h-1}})^{q_{h-1}}((p^t)^{q_h+\dots+q_m} - (p^{t-1})^{q_h+\dots+q_m})}{(p^t - p^{t-1})}$$

complete symmetric graphs of order $(p^t - p^{t-1})$, for $t = 1, 2, \dots w_m$. When $t = 0$, the identity forms a complete symmetric graph of order 1. Lemma 16 yields the formula in (iii) and completes the proof. □

Chapter 4

Cayley D-saturated semigroups

This chapter is devoted to Cayley graphs of semigroups. Let S be a semigroup, and let T be a non-empty subset of S. The *Cayley graph* $\mathrm{Cay}(S,T)$ of S with respect to T is defined as the graph with vertex set S and edge set $E(S)$ consisting of those ordered pairs (x,y) where $x \neq y$ and $xt = y$, for some $t \in T$ (see [57]). We refer to [26] and [28] for earlier results on Cayley graphs of semigroups.

A semigroup S is *Cayley D-saturated* with respect to a set T of S if, for each infinite subset V of S, there exists a subgraph of $\mathrm{Cay}(S,T)$ that is isomorphic to D with all vertices being in V. Evidently, if D is null, then all semigroups S are Cayley D-saturated with respect to all infinite subsets T of S.

Also, if T is finite, then the outdegree of every vertex in $\mathrm{Cay}(S,T)$ is finite. Therefore $\mathrm{Cay}(S,T)$ does not contain a copy of A_∞, D_∞ or K_∞. Hence $\mathrm{Cay}(S,T)$ contains an infinite null subgraph, by Corollary 21, and so S is not Cayley D-saturated with respect to T, unless D is null.

We begin by showing that the class of Cayley D-saturated semigroups is closed under subsemigroups and quotient groups, but not direct products.

The first main theorem describes all pairs (D,S), where D is a finite graph with at least one edge and S is a semigroup such that S is Cayley D-saturated with respect to S. The next theorem demonstrates that, for each finite graph D, every infinite semigroup S contains an infinite subset T such that S is not D-saturated with respect to T.

We then describe several classes of semigroups whose Cayley graphs belong to one the classes well known in graph theory. All finite inverse semigroups with bipartite Cayley graphs are characterised. This leads to a description of all commutative inverse semigroups with bipartite Cayley graphs. Next, algebraic conditions are given for semigroups with Cayley graphs that are disjoint unions of complete graphs. This result is used to describe all monoids S and subsets T of S such that $\mathrm{Cay}(S,T)$ is isomorphic to the disjoint union of complete graphs.

These results have been submitted for publication in [30] and [33].

4.1 Properties of Cayley D-saturated semigroups

Lemma 33 *Let D be a finite graph, S a semigroup, U a subsemigroup of S, and let T be a subset of U. If S is Cayley D-saturated with respect to T, then U is Cayley D-saturated with respect to T.*

Proof. Take any infinite subset $T \subseteq U$. By the definition of a Cayley graph, $\mathrm{Cay}(U,T)$ is isomorphic to the subgraph G of $\mathrm{Cay}(S,T)$ induced by the vertices of U. Since D embeds in G, it embeds in $\mathrm{Cay}(U,T)$. Therefore U is Cayley D-saturated. □

If T is not contained in U, then it is not true in general that U is D-saturated with respect to $T \cap U$, as seen in the next example.

Example 34 Let G be an infinite group, $I = \{0,1\}$, and let S be the semigroup with zero θ and all elements of $G \times I$, where multiplication is defined by the rule

$$(g,i)(h,j) = \begin{cases} (gh, ij) & \text{if } i = 1 \text{ or } j = 1, \\ \theta & \text{if } i = j = 0. \end{cases}$$

The semigroup S is Cayley D-saturated with respect to $T = S$. The subset $U = G \times \{0\} \cup \theta$ is a subsemigroup of S, but U is not Cayley D-saturated with respect to $T \cap U = U$, since the elements of U induce a null subgraph in $\mathrm{Cay}(U,U)$.

Lemma 35 *Let S be a Cayley D-saturated semigroup with respect to T, and let D be a finite graph. Let ρ be a congruence on S. Then S/ρ is finite if and only if ρT is finite.*

Proof. If ρT is infinite, then obviously S/ρ is infinite.

Suppose that S/ρ is infinite and that ρT is finite. Take any infinite subset V of S/ρ and, for each $v_i \in V$, choose a representative $s_i \in S$ such that $\rho s_i = v_i$. Let W be the set of these representatives. Then $\rho^\natural : W \to V$ is a bijection, and so W is infinite.

Since S is Cayley D-saturated, $\mathrm{Cay}(S,T)$ contains a subgraph G isomorphic to A_∞, D_∞ or K_∞ with all vertices being in W, by Corollary 21. Pick any element s_i in G with outdegree greater than $|\rho T|$. Then there exist $s_j, s_k \in G$ such that $s_i t = s_j$ and $s_i t' = s_k$, where $t, t' \in T$ and $\rho t = \rho t'$. Thus

$$v_k = \rho s_k = \rho(s_i t') = \rho s_i \rho t' = \rho s_i \rho t = \rho(s_i t) = \rho s_j = v_j.$$

This contradicts the fact that $\rho^\natural$ is a bijection, and completes the proof. □

Lemma 36 *Let S be a Cayley D-saturated semigroup with respect to T, D a finite graph, and let ρ be a congruence on S. Then S/ρ is Cayley D-saturated with respect to ρT.*

Proof. If S/ρ is finite, then it vacuously true that S/ρ is Cayley D-saturated with respect to ρT, and there is nothing to prove.

Suppose that S/ρ is infinite. Let $\rho^\natural$ be the bijection $\rho^\natural : W \to V$, where V and W are defined in Lemma 35. Then W is infinite, and so D is contained in the subgraph G of $\mathrm{Cay}(S,T)$ with all vertices being in W.

If $(s_i, s_j) \in E(D)$, then $s_i t = s_j$, for some $t \in T$. Therefore

$$v_i \rho t = \rho s_i \rho t = \rho(s_i t) = \rho s_j = v_j,$$

and so $(v_i, v_j) \in \mathrm{Cay}(S/\rho, \rho T)$. Thus D embeds in $\mathrm{Cay}(S/\rho, \rho T)$, and so S/ρ is D-saturated with respect to ρT. □

Example 37 shows that the class of all Cayley D-saturated semigroups is not closed under direct products.

Example 37 Let G be any infinite group, and let

$$S = \{a, b \mid a^2 = ab = ba = b^2 = b\}.$$

Evidently, $\mathrm{Cay}(G, G) \cong K_\infty$, and all finite semigroups are vacuously D-saturated. For any subset T of $G \times S$, there are no edges in $\mathrm{Cay}(G \times S, T)$, between vertices of the set $\{(g, a) \mid g \in G\}$. Therefore $G \times S$ is not Cayley D-saturated with respect $G \times S$.

4.2 Cayley D-saturated semigroups when $T = S$

The next theorem shows that when $T = S$, there are nontrivial sufficient and necessary conditions for when a semigroup S is D-saturated.

Theorem 38 *Let D be a finite graph which is not null, and let S be an infinite semigroup. Then the following conditions are equivalent:*

(i) *S is Cayley D-saturated with respect to S;*

(ii) *S does not have an infinite set of pairwise incomparable principal right ideals, and if D has a cycle, then the number of principal right ideals in S is finite.*

Proof. (i)$\Rightarrow$(ii): Suppose to the contrary that there exists an infinite set of pairwise incomparable principal right ideals in S. Then the elements generating these ideals induce a null subgraph in $\mathrm{Cay}(S, S)$. Then D does not embed in $\mathrm{Cay}(S, S)$, which contradicts (i).

Suppose that S has infinitely many principal right ideals. These ideals induce a partial order in $S/\mathcal{R}$. Then the graph G with all elements of $S/\mathcal{R}$ as vertices and edges (R_u, R_v), for all $R_u, R_v \in S/\mathcal{R}$ and $R_u < R_v$ is infinite. Hence Corollary 21 implies that there exists an infinite subset of elements of $S/\mathcal{R}$ which induces a subgraph in G that is either null, or isomorphic to an infinite descending chain, or isomorphic to an infinite ascending chain, or isomorphic to an infinite complete symmetric graph. In the first case S contains an infinite set of pairwise incomparable principal right ideals, which has been precluded. The last case is excluded since the G is acyclic. In the second and third cases the infinite ascending (resp., descending) chain in G is induced by a set of elements $R_{s_1}, R_{s_2}, \ldots$ which forms an infinite ascending (resp., descending) chain in $S/\mathcal{R}$. Then the elements $s_1, s_2, \ldots$ induce a subgraph in $\mathrm{Cay}(S,S)$ which is isomorphic to an infinite descending (resp., ascending) chain. The graph D embeds in this subgraph, and so is acyclic.

(ii)$\Rightarrow$(i): First, suppose that D has a cycle. Take an infinite set V of S. Since S has a finite number of right ideals, there exists an infinite subset U of V whose elements lie in same $\mathcal{R}$-class of S. Clearly, the elements of U induce an infinite complete symmetric graph in $\mathrm{Cay}(S,S)$, and D embeds in this subgraph.

Second, consider the case where D has no cycles. Then D embeds in A_∞ or D_∞, by Lemma 14. Take any infinite set V of S. Given that S has no infinite set of pairwise incomparable principal right ideals, Corollary 21 implies that there exists an infinite sequence of elements U in V generating principal right ideals which forms an infinite ascending chain or an infinite descending chain in $S/\mathcal{R}$. Then the vertices of U induce a subgraph in $\mathrm{Cay}(S,S)$ that contains an infinite descending chain or an infinite ascending chain, respectively. The graph D embeds in these subgraphs, and so S is D-saturated with respect to S. $\square$

The question arises as to whether a semigroup is Cayley D-saturated if no restrictions are placed on T. The next sections show that, for every finite graph D with edges and every infinite semigroup S, there exists an infinite subset T of S such that S is not Cayley D-saturated with respect to T.

4.3 Groups

Lemma 39 *Let D be a finite graph, and let G be an infinite group with identity e. Then there exists an infinite subset T of G such that G is not D-saturated with respect to T.*

Proof. We use the fact that if $T \subseteq G$ and $T^2 \cap T = \emptyset$, then the vertices of T induce a null subgraph in $\mathrm{Cay}(G,T)$. We will show by induction that there always exists an infinite set T of G such that $T^2 \cap T = \emptyset$.

Suppose that G contains an element g and an infinite subset H_g such that $h^2 = g$, for all $h \in H_g$. Pick two elements $h_1, h_2 \in H_g \setminus \{e\}$, and let $T_2 = \{h_1, h_2\}$. Then $T_2^2 \cap T_2 = \emptyset$. Assume that there exists a set $T_n \subset H_g \setminus \{e\}$ of n elements such that $T_n^2 \cap T_n = \emptyset$. Put $T_n^{-1} = \{h \in G \mid h^{-1} \in T_n\}$. Then the set

$$Q = T_n T_n^{-1} \cup T_n^{-1} T_n \cup T_n^2$$

is finite, and so $H_g \setminus \{\{H_g \cap Q\} \cup e\}$ is infinite. Take any $h \in H_g \setminus \{\{H_g \cap Q\} \cup e\}$. Then

$$hT_n \cap \{T_n \cup h\} = T_n h \cap \{T_n \cup h\} = h \cap T_n^2 = h^2 \cap T_n = \emptyset,$$

and so the set $T_{n+1} = \{T_n \cup h\}$ satisfies $T_{n+1}^2 \cap T_{n+1} = \emptyset$. Applying the induction hypothesis we see that in this case there exists an infinite set T such that $T^2 \cap T = \emptyset$.

If $H_g = \{h \in G \mid h^2 = g\}$ is finite, for all $g \in G$, then pick two elements $g_1 \in G \setminus \{e\}$ and $g_2 \in G \setminus \{H_{g_1} \cup g_1^2 \cup e\}$. The set $T_2 = \{g_1, g_2\}$ satisfies $T_2^2 \cap T_2 = \emptyset$. Assume that there exists a set $T_n \subset G$ of n elements such that $T_n^2 \cap T_n = \emptyset$. Put $T_n^{-1} = \{g \in G \mid g^{-1} \in T_n\}$. It follows that the set

$$Q = T_n T_n^{-1} \cup T_n^{-1} T_n \cup T_n^2$$

is finite. The set

$$R = \{H_{g_1} \cup H_{g_2} \cup \ldots \cup H_{g_n} \cup g_1^2 \cup g_2^2 \cup \ldots g_n^2 \cup e\}$$

is also finite, and so $G \setminus \{Q \cup R\}$ is infinite. Take any $g \in G \setminus \{Q \cup R\}$. Then

$$gT_n \cap \{T_n \cup g\} = T_n g \cap \{T_n \cup g\} = g \cap T_n^2 = g^2 \cap T_n = \emptyset,$$

and so the set $T_{n+1} = \{T_n \cup g\}$ satisfies $T_{n+1}^2 \cap T_{n+1} = \emptyset$. Applying the induction hypothesis we see that in this case too there exists an infinite set T such that $T^2 \cap T = \emptyset$. Hence the vertices of T are not adjacent in $\mathrm{Cay}(G, T)$. This implies that $\mathrm{Cay}(G, T)$ does not contain a subgraph isomorphic to D, unless D is null. □

4.4 Nil semigroups

Next, we show that there are no infinite nil semigroups S that are Cayley D-saturated with respect to all infinite subsets T of S. The proof relies on the following result due to L. N. Shevrin.

Lemma 40 ([52], Theorem 1.4.3) *If all nilpotent subsemigroups of a nilsemigroup S are finite, then S is finite.*

Lemma 41 *Let D be a finite graph with at least one edge, and let S be an infinite nilsemigroup. Then there exists an infinite subset T of S such that S is not Cayley D-saturated with respect to T.*

Proof. Lemma 40 shows that S contains an infinite nilpotent subsemigroup, say W. Suppose that $W^n = 0$, for some positive integer n. If $W \setminus W^2$ is finite, then $W = \cup_{k=1}^{n}(W \setminus W^2)^k$ is finite, a contradiction. Thus $W \setminus W^2$ is infinite. Partition $W \setminus W^2$ into two infinite subsets T and U. Then $UT \cap U = \emptyset$, and hence the subgraph G of $\mathrm{Cay}(W, T)$ induced by the vertices of U is null. Therefore W is not D-saturated with respect to T. Thus S is not Cayley D-saturated with respect to T, by Lemma 33. □

4.5 Arbitrary semigroups

The structure of unipotent epigroups is described by the following lemma.

Lemma 42 ([52], Proposition 1.5.2) *A semigroup is a unipotent epigroup if and only if it is an ideal extension of a group by a nilsemigroup.*

The following results are also needed for the proof of our next main theorem.

Lemma 43 ([52], Proposition 11.9) *Every epigroup with infinitely many idempotents has a subsemigroup with a unique infinite basis.*

Lemma 44 ([52], Corollary 11.7) *An infinite epigroup having finitely many idempotents contains an infinite unipotent subepigroup.*

Theorem 45 *Let D be a finite graph which is not null. Each infinite semigroup S has an infinite subset T such that S is not D-saturated with respect to T.*

Proof. Suppose to the contrary that D is a graph which is not null and that S is Cayley D-saturated with respect to all infinite subsets T of S.

If S has an element s which is not periodic, then the vertices $s^2, s^4, \ldots$ are not adjacent in $\mathrm{Cay}(S,T)$, where $T = \{s, s^3, s^5, \ldots\}$. Therefore D does not embed in the subgraph induced by the elements $s^2, s^4, s^6, \ldots$, a contradiction. Thus S is periodic, and hence an epigroup.

If S contains infinitely many idempotents, then Lemma 43 implies that S contains a subsemigroup W with an infinite basis X. Partition X into two infinite subsets T and U. It follows that $UT \cap U = \emptyset$, and so the subgraph of $\mathrm{Cay}(S,T)$ with vertex set U is null. Hence S is not Cayley D-saturated with respect to T in this case.

Therefore S contains a finite number of idempotents. Denote all of them by $e_1, e_2, \ldots e_n$. By Lemma 44 there exists an infinite unipotent subepigroup S_{e_i} for at least one value of $1 \leq i \leq n$. We know by Lemma 42 that S_{e_i} is an ideal extension of a group by a nilsemigroup.

If the group G_{e_i} is infinite, then it follows from Lemma 39 that there exists a subset T of G_{e_i} such that G_{e_i} is not Cayley D-saturated with respect to T. Lemma 33 shows that K_{e_i}, and hence S is not Cayley D-saturated with respect to the same infinite subset T of G_{e_i}.

If, however, G_{e_i} is finite , then the nilextension S_{e_i}/G_{e_i} is infinite. Lemma 41 shows that S_{e_i}/G_{e_i} is not D-saturated with respect to some infinite subset T of S_{e_i}/G_{e_i}. Then Lemma 36 tells us that S_{e_i} is not Cayley D-saturated with respect to some infinite subset T' of S_{e_i}, where the image of T' is T in S_{e_i}/G_{e_i}. Therefore S is not D-saturated with respect to T', by Lemma 33. □

4.6 Inverse semigroups with bipartite Cayley graphs

In this section we describe all finite inverse semigroups and all commutative inverse semigroups whose Cayley graphs are bipartite.

We need the following well-known lemma and include a proof for completeness.

Lemma 46 *The kernel of a finite inverse semigroup S is a group.*

Proof. The kernel K of S exists, since S is finite. If S has a zero, then the result is trivial. Assume that $0 \notin S$. Since K is an ideal, K is an inverse subsemigroup of S. By [20], Corollary 2.4.10, K is completely simple. If K contains two idempotents e, f, then $ef = fe = 0$, since every $\mathcal{L}$-class and every $\mathcal{R}$-class of K contains exactly one idempotent, by [21], Theorem 5.1.1. This contradiction shows that K contains exactly one idempotent. A completely simple semigroup containing one idempotent is a group. $\Box$

The following lemma is also required for our proof.

Lemma 47 *Let S be a semigroup with an ideal K that is a group, and let T be a non-empty subset of S. Then the following conditions are equivalent:*

(i) *the Cayley graph* $\mathrm{Cay}(S,T)$ *is bipartite;*

(ii) *K has a subgroup H such that there exists a normal subgroup N of H with index 2, and $eT \subseteq H \setminus N$, where e is the identity of K.*

Proof. For any $s \in S$ the element es belongs to K, since K is an ideal. Let $H = \langle eT \cup T^{-1} \rangle$, where $T^{-1} = \{k \in K \mid k = (et)^{-1}$, for some $t \in T\}$. For each $h = h_1 h_2 \dots h_n \in H$, the element $h^{-1} = h_n^{-1} h_{n-1}^{-1} \dots h_1^{-1} \in H$, by [50] 1.1.4. Therefore H is a subgroup of K, by Lemma 1.

(i)$\Rightarrow$(ii): Suppose that $\mathrm{Cay}(S,T)$ is bipartite, that is there exists a partition $S = S_0 \cup S_1$ such that

$$E(\mathrm{Cay}(S,T)) \subseteq (S_0 \times S_1) \cup (S_1 \times S_0).$$

We may assume that $e \in S_0$, and that $e \notin T$, since $\mathrm{Cay}(S,T)$ does not contain loops. If $t \in T \cup T^{-1}$, then $(e, et) \in E(\mathrm{Cay}(S,T))$. Therefore the set $e(T \cup T^{-1}) \subseteq S_1$. Induction on k shows that $e(T \cup T^{-1})^k \subseteq S_0$, for all even k, and $e(T \cup T^{-1})^k \subseteq S_1$, for all odd k. Put $e(T \cup T^{-1})^0 = \{e\}$, $N = \cup_{k=0}^{\infty} e(T \cup T^{-1})^{2k}$ and $M = \cup_{k=0}^{\infty} e(T \cup T^{-1})^{2k+1}$. Clearly, N is a subgroup of $H = N \cup M$ and $eT \subseteq e(T \cup T^{-1}) \subseteq M = H \setminus N$.

For all $h \in H$, we know that $eh = he = h$, since e is the identity of K and $H \subset K$. Assume that $he(T \cup T^{-1})^{2k} = e(T \cup T^{-1})^{2k} h$, for all $1 \le k < n$.

Then

$$
\begin{aligned}
& he(T\cup T^{-1})^{2n} = he(T\cup T^{-1})^{2n-2}(T\cup T^{-1})^2 \\
= \ & e(T\cup T^{-1})^{2n-2}h(T\cup T^{-1})^2 = e(T\cup T^{-1})^{2n-2}he(T\cup T^{-1})^2 \\
= \ & e(T\cup T^{-1})^{2n-2}e(T\cup T^{-1})^2h = e(T\cup T^{-1})^{2n}h.
\end{aligned}
$$

Easy induction shows that $heT^{2k} = eT^{2k}h$, for all $k \geq 1$. Hence $hN = Nh$, and so N is a normal subgroup of H. Clearly, N has index 2 in H. Thus (ii) holds.

(ii)$\Rightarrow$(i): Suppose that the kernel K of S has a subgroup H such that there exists a normal subgroup N of H with index 2, and $eT \subseteq M = H \setminus N$. Choose elements r_i, $i \in I$, in K such that K is a disjoint union of the left cosets r_iH. Put

$$K_0 = \cup_{i\in I} r_iN, \quad S_0 = \{s \in S \mid es \in K_0\},$$

$$K_1 = \cup_{i\in I} r_iM, \quad S_1 = \{s \in S \mid es \in K_1\}.$$

Then $K = K_0 \cup K_1$ and $S = S_0 \cup S_1$. We get

$$e(S_0T) = eS_0eT \subseteq eS_0M \subseteq K_0M = \cup_{i\in I} r_iNM = \cup_{i\in I} r_iM = K_1,$$

and so the definition of S_1 shows that

$$S_0T \subseteq S_1. \tag{4.1}$$

Similarly,

$$e(S_1T) = eS_1eT \subseteq eS_1M \subseteq K_1M = \cup_{i\in I} r_iMM = \cup_{i\in I} r_iN = K_0,$$

and hence

$$S_1T \subseteq S_0. \tag{4.2}$$

Inclusions (4.1) and (4.2) show that $\mathrm{Cay}(S,T)$ is a bipartite graph. □

If K is a torsion group, then it is not difficult to see that $\langle eT\rangle$ is a subgroup of K. The following example shows that this is not necessarily the case when K is not torsion.

Example 48 Let $S = (\mathbb{Z}, +)$ and $T = \{1\}$. Then

$$H = \mathbb{Z}, \quad N = \{t \in S \mid t = 2s, s \in \mathbb{Z}\},$$

and S is bipartite with partition $S = N \cup S \setminus N$. The set $\langle eT\rangle = \mathbb{Z}^+$ is not a subgroup of H.

Theorem 49 *For every finite inverse semigroup S and non-empty subset T of S, the following conditions are equivalent:*

(i) *the Cayley graph* $\mathrm{Cay}(S,T)$ *is bipartite;*

(ii) *the kernel K of S has a subgroup H such that there exists a normal subgroup N of H with index 2, and $eT \subseteq H \setminus N$, where e is the identity of the group K.*

Proof. The result follows from Lemma 47, because the kernel of every finite inverse semigroup is a group, by Lemma 46. □

The following example of a finite inverse semigroup with bipartite Cayley graph illustrates Theorem 49.

Example 50 Let $S = \{e, s, (1,1), (1,2), (2,1), (2,2)\}$ be a set with multiplication defined by

	e	s	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
e	e	s	e	s	s	e
s	s	e	s	e	e	s
$(1,1)$	e	s	$(1,1)$	$(1,2)$	s	e
$(1,2)$	s	e	s	e	$(1,1)$	$(1,2)$
$(2,1)$	s	e	$(2,1)$	$(2,2)$	e	s
$(2,2)$	e	s	e	s	$(2,1)$	$(2,2)$.

Then S is a semigroup with kernel $K = \{e, s\}$. Let $T = \{s, (1,2), (2,1)\}$. Since $N = \{e\}$ is a normal subgroup of index 2 in K such that $eT \subseteq K \setminus N$, the graph $\mathrm{Cay}(S,T)$ is bipartite, by Theorem 49.

Theorem 51 *For every commutative inverse semigroup S with a non-empty subset T, the following conditions are equivalent:*

(i) *the Cayley graph* $\mathrm{Cay}(S,T)$ *is bipartite;*

(ii) *for each idempotent e of S, the group G_e has a subgroup H_e with a normal subgroup N_e such that $(eT \cap G_e) \subseteq H_e \setminus N_e$ and N_e has index 2 in H_e.*

Proof. (i)⇒(ii): Every commutative inverse semigroup is a semilattice of groups, by [21], Theorem 4.2.1. For each idempotent $e \in S$, let

$$G^{(e)} = \{s \in S \mid es \in G_e\}.$$

If $s \in G^{(e)}$, then $s^{-1} \in G^{(e)}$, since $es^{-1} \in G_e G_{s^{-1}} = G_e G_s \subseteq G_{es} = G_e$. Therefore $G^{(e)}$ is regular. All idempotents commute since S is commutative, and so $G^{(e)}$ is an inverse semigroup with kernel G_e, by [21], Theorem 5.1.1. Then the result follows from Lemma 47.

(ii)⇒(i): Suppose that for each idempotent e of S, the group G_e has a subgroup H_e with a normal subgroup N_e of index 2 in H_e such that $(eT \cap G_e)$ is a subset of $H_e \setminus N_e$. Put $M_e = H_e \setminus N_e$.

By [11], Theorem 1.9, a graph is bipartite if and only if it has no cycles of odd length. If $\mathrm{Cay}(S,T)$ has a cycle $s_0, s_1, \ldots, s_{2n+1} = s_0$ of odd length, then $s_i t_i = s_{i+1}$, for some $t_i \in T$ and $0 \le i \le 2n-1$, and $s_{2n}t_{2n} = s_0$, for some $t_{2n} \in T$. For $i = 0, \ldots, 2n$, there exist idempotents $e_i, f_i \in S$ such that $s_i \in G_{e_i}$ and $t_i \in G_{f_i}$. Put $e = e_0 e_1 \ldots e_{2n} f_0 f_1 \ldots f_{2n}$. Since $G^{(e)}$ is an inverse semigroup with kernel G_e, Lemma 47 tells us that the Cayley graph $\mathrm{Cay}(G^{(e)}, T \cap G^{(e)})$ is bipartite. However, all elements $s_0, s_1, \ldots, s_{2n}$ and $t_0, t_1, \ldots, t_{2n}$ belong to $G^{(e)}$. Therefore the graph $\mathrm{Cay}(G^{(e)}, T \cap G^{(e)})$ has a cycle of odd length. This contradiction completes the proof. □

The next example shows that neither Theorem 49 nor Theorem 51 generalise to arbitrary inverse semigroups.

Example 52 Let S be the bicyclic monoid with generators a, b and relation $ab = 1$. Then S is an infinite inverse non-commutative semigroup with no kernel. Let T be the set of all elements $b^m a^n$ of S, where m, n are nonnegative integers such that $m+n$ is odd. The graph $\mathrm{Cay}(S,T)$ is bipartite with partition $S = T \cup S \setminus T$.

4.7 Semigroups with Cayley graphs that are disjoint unions of complete graphs

In this section we describe all semigroups with Cayley graphs that are disjoint unions of complete graphs. This result is used to characterise all inverse semigroups with the same property.

Theorem 53 *For every semigroup S with a non-empty subset T, the Cayley graph* $\mathrm{Cay}(S,T)$ *is a disjoint union of complete graphs if and only if the following conditions hold:*

(i) $g, gt \in gsT^1$, *for all* $s, t \in T, g \in S$;

(ii) $gT^1 = g(T^1)^2$, *for all* $g \in S$.

Proof. The 'only if' part. Take any elements $s, t \in T, g \in S$. If $gs = g$ and $gs = gt$, then there is nothing to prove. Therefore we may assume that $gs \neq g, gt$. By the definition of a Cayley graph, (g, gs) and (gs, gt) are edges of $\mathrm{Cay}(S,T)$. Hence there exists a complete subgraph of $\mathrm{Cay}(S,T)$ containing g, gs and gt. Thus (gs, g) and (gs, gt) are also edges of this complete graph, and so there exist $u, v \in T$ such that $g = gsu$ and $gt = gsv$, respectively. Therefore $g, gt \in gsT^1$, that is, (i) holds.

It is clear that the set gT^1 is precisely the set of vertices of the connected component of the Cayley graph $\mathrm{Cay}(S,T)$ containing g, and that $g(T^1)^2$ is the same connected component. Therefore (ii) holds.

The 'if' part. Suppose that $gT^1 = g(T^1)^2$ and $g, gt \in gsT^1$, for all $s, t \in T$, $g \in S$.

Take any element $h \in S$, and consider the subgraph $\mathrm{Cay}(hT^1, T)$ induced by the set hT^1 in the Cayley graph $\mathrm{Cay}(S,T)$. For any two distinct vertices $hs, ht \in hT^1$, we know that $ht \in hsT$, and so (hs, ht) is an edge of $\mathrm{Cay}(hT^1, T)$. Therefore $\mathrm{Cay}(hT^1, T)$ is a complete graph.

Suppose that two sets xT^1 and yT^1 have a common element $z = xs = yt$, where $x, y, z \in S$, $s, t \in T^1$. Take any element $xu \in xT^1$, where $u \in T^1$. We know that $xu \in xsT^1$ by (i), and so $xu = xsw$, for some $w \in T^1$. Hence $xu = xsw = ytw \in y(T^1)^2 = yT^1$, by (ii). Therefore $xT^1 \subseteq yT^1$. Similarly, $xT^1 \supseteq yT^1$, and so $xT^1 = yT^1$. Thus $\mathrm{Cay}(S,T)$ is a disjoint union of complete graphs. $\square$

Whilst this result applies to semigroups in general, it does not provide a clear picture of the structure of semigroups possessing this property. Nevertheless, it is a useful tool in characterising all monoids with Cayley graphs that are disjoint unions of complete graphs.

Corollary 54 *For every monoid S with a non-empty subset T, the following conditions are equivalent:*

(i) *the Cayley graph* $\mathrm{Cay}(S,T)$ *is a disjoint union of complete graphs;*

(ii) T^1 *is a subgroup of the group of units of* S.

Proof. (i)$\Rightarrow$(ii): Suppose that $\mathrm{Cay}(S,T)$ is a union of complete graphs.

Letting $g = 1$ in Theorem 53 (ii), we see that $T^1 = (T^1)^2$. That is, T^1 is a monoid. We also know from Theorem 53 (i) that $1 \in sT^1$ for all $s \in T$. Therefore to each element $s \in T$ there corresponds an element $t \in T$ such that $st = 1$. This implies that T^1 is a group.

(ii)$\Rightarrow$(i): For any $x \in S$, consider the subgraph $\mathrm{Cay}(xT^1,T)$ of the Cayley graph $\mathrm{Cay}(S,T)$, induced by the set xT^1 of vertices. For any $s,t \in T$, we have $xs(s^{-1}t) = xt$, since T^1 is a group. Therefore (xs,xt) is an edge of $\mathrm{Cay}(xT^1,T)$, and so $\mathrm{Cay}(xT^1,T)$ is a complete graph.

Suppose that, for some $x,y \in S$, the cosets xT^1 and yT^1 intersect. Fix any $z \in xT^1 \cap yT^1$. There exist $s,t \in T^1$ such that $z = xs = yt$. For each $w \in yT^1$, where $w = yu$, $u \in T^1$, we get $w = yu = yt(t^{-1}u) = xs(t^{-1}u) \in xT^1$. Hence $yT^1 \subseteq xT^1$. Similarly, $yT^1 \supseteq xT^1$, and therefore these two cosets coincide.

We have shown that the connected components of $\mathrm{Cay}(S,T)$ are precisely the complete graphs $\mathrm{Cay}(sT^1,T)$ induced by the left cosets sT^1, where $s \in S$. Thus $\mathrm{Cay}(S,T)$ is a disjoint union of complete graphs. □

The next example shows that it is not necessary that S be a monoid for $\mathrm{Cay}(S,T)$ to be disjoint union of complete graphs.

Example 55 Let X be a countably infinite set, and let S be the Baer-Levi semigroup of one-to-one maps $f : X \to X$ such that $X \setminus (X)f$ is infinite. Then S is right simple and has no idempotents, by [13], Theorem 8.2. Take any element $x \in S$. The right ideal xS^1 generated by x coincides with S. Therefore there exists $s \in S$ such that $xs = g$, for each $g \in S$. This means that (x,g) is an edge of the Cayley graph $\mathrm{Cay}(S,S)$. Thus $\mathrm{Cay}(S,S)$ is a complete graph.

Chapter 5

Divisibility D-saturated semigroups

This chapter is devoted to a combinatorial property defined in terms of divisibility graphs. The *divisibility graph* $\mathrm{Div}(S)$ of a semigroup S has all elements of S as vertices and all edges (u, v), where $u \neq v$ and u *divides* v, i.e., $u = avb$ for $a, b \in S^1$. For a finite graph D, a semigroup is *divisibility D-saturated* if and only if, for every infinite subset T of S, the graph $\mathrm{Div}(S)$ has a subgraph isomorphic to D with all vertices being in T.

If U is a subsemigroup of S and T a subset of U, then superscripts are used to avoid ambiguity about the meaning of a divisibility graph. Thus $\mathrm{Div}(T)^U$ (resp., $\mathrm{Div}(T)^S$) denotes the divisibility graph induced by the elements of T in U (resp., S). That is, $(a, b) \in E(\mathrm{Div}(T)^U)$ (resp., $E(\mathrm{Div}(T)^S)$) if and only if $a \neq b$ and $a = xby$, for some $x, y \in U^1$ (resp., $x, y \in S^1$).

First, it is shown that the class of all divisibility D-saturated semigroups is closed under homomorphic images, but not under subsemigroups or direct products. Then, necessary and sufficient conditions are given for when a commutative semigroup is divisibility D-saturated. We describe all subepigroups of completely 0-simple semigroups that are divisibility D-saturated. This result is used to characterise all monomial matrix epigroups with the same property.

5.1 Properties of divisibility D-saturated semigroups

If S is a group, then $\mathrm{Div}(S)$ is isomorphic to a complete symmetric graph of cardinality $|S|$, and so all infinite groups are divisibility D-saturated, for all finite graphs D. In view of Theorem 29, it is clear the all power D-saturated semigroups are divisibility D-saturated. The converse is not true, since not all groups are power D-saturated.

The first lemma shows that any homomorphic image of a D-saturated semigroup is D-saturated.

Lemma 56 *Let S be any D-saturated semigroup, and let ρ be a congruence on S. Then S/ρ is D-saturated.*

Proof. If S/ρ is finite then the result is vacuously true. Therefore we may assume that S/ρ is infinite.

Take any infinite subset $T \subseteq S/\rho$. For each $t_i \in T$, choose a representative $s_i \in S$ such that $\rho s_i = t_i$, and denote by U the set of all of these representatives. The map $\rho^\natural : U \to T$ is a bijection, and so U is infinite. Therefore D embeds in the subgraph G of $\mathrm{Div}(S)$ with all vertices being in U, since S is D-saturated.

If $(s_i, s_j) \in E(G)$, then $s_i = as_jb$ for some $a, b \in S^1$. Therefore

$$t_i = \rho s_i = \rho(as_jb) = \rho a \rho s_j \rho b = \rho a t_j \rho b,$$

and so $(t_i, t_j) \in E(\mathrm{Div}(S/\rho))$. Thus G embeds in $\mathrm{Div}(S/\rho)$, and so D embeds in $\mathrm{Div}(S/\rho)$. Hence S/ρ is D-saturated. □

The class of all divisibility D-saturated semigroups happens to be more complicated than those of power D-saturated semigroups and Cayley D-saturated semigroups. In particular, it is not closed under subsemigroups as seen in Example 34. In this example S has 3 ideals, namely $\theta, G \times \{0\} \cup \theta$ and S, and so it is divisibility D-saturated, for all graphs D. However, the subsemigroup $G \times \{0\} \cup \theta$ contains an infinite null graph, and so it is D-saturated only if D is a null graph.

Thus divisibility D-saturated semigroups do not form a variety.

Example 34 also shows that infinitely many elements of a divisibility D-saturated semigroup S may belong to $\mathcal{H}$-classes of S which are not subgroups. Again, this is in contrast to the case of power D-saturated semigroups.

5.2 The cardinality of $S/\mathcal{J}$ and cycles in D

The question as to whether a semigroup S is divisibility D-saturated is fundamentally concerned with the structure of $S/\mathcal{J}$, and whether D contains a cycle or not. The characterization of divisibility D-saturated semigroups in this section is analogous to the characterization of Cayley D-saturated semigroups given Theorem 38 in Section 4.2, since $\mathrm{Div}(S)$ relates to the Green equivalence $\mathcal{J}$ precisely as $\mathrm{Cay}(S, S)$ relates to $\mathcal{R}$. For the convenience of the reader, we include self-contained proofs of the next two results.

We remark that the partially ordered set of principal ideals for any semigroup S is isomorphic to the partial order $S/\mathcal{J}$.

First, the case when D is acyclic is considered.

Lemma 57 *Let S be a semigroup, and let D be a finite acyclic graph that has at least one edge. Then the following are equivalent:*

(i) *S is divisibility D-saturated;*

(ii) *the partially ordered set of principal ideals of S does not contain infinite antichains.*

Proof. (i)$\Rightarrow$(ii): Suppose to the contrary that there exists an infinite antichain of principal ideals $J(x_1), J(x_2), \dots$ in S. If $(x_i, x_j) \in E(\mathrm{Div}(S))$, then $x_i = ax_jb$, for some $a, b \in S^1$. It follows that $J(x_i) \leq J(x_j)$, a contradiction. Therefore the vertices $x_1, x_2, \dots$ induce an infinite null subgraph in $\mathrm{Div}(S)$, which contradicts (i).

(ii)$\Rightarrow$(i): Take any infinite set T of S. By Corollary 21, the subgraph G of $\mathrm{Div}(S)$ with all vertices being in T contains an infinite null subgraph, or an infinite ascending chain, or an infinite descending chain, or an infinite complete symmetric graph.

If G has an infinite null subgraph, then we get a contradiction with (ii), and so this case is impossible.

In the last case, where G contains an infinite complete symmetric graph, it is clear that D embeds in G.

Suppose that G contains a subgraph isomorphic to A_∞ or D_∞. Every acyclic graph D embeds in a chain with the same set of vertices, by Lemma 14. Hence G has a subgraph isomorphic to D. Thus (i) holds. □

The next result applies when D contains a cycle.

Lemma 58 *Let S be a semigroup, and let D be a finite graph with a cycle. Then the following are equivalent:*

(i) *S is divisibility D-saturated;*

(ii) *the partially ordered set of principal ideals of S is finite.*

Proof. (i)$\Rightarrow$(ii): Suppose to the contrary that the partially ordered set of principal ideals is infinite. Denote by T the set obtained by selecting an element from each principal ideal. By Corollary 21, the subgraph G of $\mathrm{Div}(S)$ with all vertices being in T contains either an infinite null subgraph, or an infinite ascending chain, or an infinite descending chain, or an infinite complete symmetric graph.

The first case contradicts (i), and so is impossible.

By the definition of T, the graph G is acyclic, and so G contains a subgraph isomorphic to an infinite ascending chain or an infinite descending chain. Then D embeds in a chain and is acyclic; a contradiction. Hence (ii) follows.

(ii)$\Rightarrow$(i): Take any infinite subset T of S. By (ii), T contains an infinite subset U such that all elements of U generate the same principal ideal. The subgraph G of $\mathrm{Div}(S)$ induced by the elements of U is isomorphic to an infinite complete symmetric graph, and so D embeds in G. Therefore D embeds in $\mathrm{Div}(S)$, and so S is D-saturated. □

In the rest of this chapter we obtain necessary and sufficient conditions for various classes of semigroups to be D-saturated.

5.3 Commutative semigroups

The main theorem in this section completely describes all pairs (D, S), where D is a finite graph and S is a divisibility D-saturated commutative semigroup. The following result is needed for the case when D contains a cycle.

Lemma 59 ([20], Lemma 4.3.1) *If S is an Archimedean semigroup without idempotents, then $xy \neq x$, for all $x, y \in S$.*

This gives:

Lemma 60 *Let S be a commutative semigroup that is a semilattice Y of Archimedean semigroups S_y. Suppose that S_x has no idempotents, for some $x \in Y$ and that $g \in S_x$. Then the principal ideals generated by $g, g^2, g^3, \ldots$ in the whole semigroup S are pairwise distinct.*

Proof. Suppose to the contrary that $S^1 g^{m+n} = S^1 g^m$. Then $g^{m+n}h = g^m$, for some $h \in S$. Hence $g^m(g^n h) = g^m$, which contradicts Lemma 59, because $g^m, g^n h \in S_x$. □

Lemma 61 *Let D be a finite graph with a cycle, and let S be a commutative semigroup that is a semilattice Y of Archimedean semigroups S_y. Denote by G_y the largest subgroup of S_y. Let ρ be the congruence of S generated by the set $\cup_{y \in Y}(G_y \times G_y)$, and let $\rho_y = \rho \cap (S_y \times S_y)$. Then the following are equivalent:*

(i) *S is divisibility D-saturated;*

(ii) *S has a finite number of idempotents, every Archimedean component S_y of S has an idempotent and S_y/ρ_y is finite;*

(iii) *the partially ordered set of principal ideals of S is finite.*

Proof. (i)$\Rightarrow$(ii): If S has infinitely many idempotents, then the subgraph generated by these idempotents in $\mathrm{Div}(S)$ is acyclic. This contradicts the D-saturation of S and shows that S has a finite number of idempotents.

Suppose that S is divisibility D-saturated. Consider an Archimedean component S_y of S and suppose that S_y has no idempotent. Take any element $g \in S_y$. By Lemma 60, the ideals generated by $g, g^2, g^3, \ldots$ in the whole S are pairwise distinct. Therefore the subgraph G induced by $\{g, g^2, \ldots\}$ in $\mathrm{Div}(S)$ is isomorphic to D_∞. Since S is D-saturated, D embeds in G, and so it is acyclic. This contradiction shows that all Archimedean components of S have idempotents.

Since $G_y \times G_y \subseteq \rho_y$, we see that S_y/ρ_y is a homomorphic image of the nil semigroup S_y/G_y, and so S_y/ρ_y is nil too. It is easily seen that $\rho = \cup_{y \in Y} \rho_y$. Therefore S/ρ is a semilattice Y of nil semigroups S_y/ρ_y.

Suppose that there are two elements of S/ρ, say s, t, which generate the same principal ideal in S/ρ. Then $s = tt'$ and $t = ss'$, for some $s', t' \in S/\rho$. Clearly, s and t belong to the same Archimedean component S_y/ρ_y. Hence $s = sh$, where $h = s't'$. There exists $x \in Y$ such that $h \in S_x/\rho_x$. Since S_x/ρ_x is nil, we get $h^n \in Y$, for some n. Hence $s = sh^n \in Y$. Similarly, $t \in Y$, and so $s = t$.

Therefore $\mathcal{J}$ is contained in ρ. Evidently $\cup_{y \in Y}(G_y \times G_y) \subseteq \mathcal{J}$, and so $\rho \subseteq \mathcal{J}^\# = \mathcal{J}$. Hence $\rho = \mathcal{J}$. Since the divisibility relation is transitive, we see that $\mathrm{Div}(S/\rho)$ has no cycles.

Next, suppose that S/ρ is infinite. For each element of S/ρ we choose one representative in S. In other words, let T be a subset of S such that $S/\rho = \{\rho t \mid t \in T\}$ and all elements of T have different images in S/ρ. Since D embeds in the subgraph H of $\mathrm{Div}(S)$ induced by the elements of T, there is a cycle in H. Evidently, the images of these elements form a cycle in $\mathrm{Div}(S/\rho)$. This contradiction shows that S/ρ is finite. (Note that S_y/G_y may be infinite, as Example 34 demonstrates.) Thus (ii) holds.

(ii)$\Rightarrow$(iii): Suppose that S has a finite number of idempotents, each Archimedean component S_y has an idempotent e_y and that S/ρ is finite. Then the semilattice Y is finite.

Since $\rho = \mathcal{J}$, the partially ordered set L of principal ideals of S is isomorphic to the partially ordered set of principal ideals of S/ρ. Different elements of S/ρ generate different ideals, and so we see that $|L| = |S/\rho|$ is finite, i.e.,

(iii) holds.

The implication (iii)$\Rightarrow$(i) follows from Lemma 58. $\square$

Theorem 62 *Let $D = (V, E)$ be a finite graph with $E \neq \emptyset$, and let S be a commutative semigroup that is a semilattice Y of Archimedean semigroups S_y. Denote by G_y the largest subgroup of S_y. Let ρ be the congruence of S generated by $\cup_{y \in Y}(G_y \times G_y)$. Then S is divisibility D-saturated if and only if one of the following conditions holds:*

(i) *D has a cycle, the number of idempotents in S is finite, each Archimedean component of S has an idempotent and S/ρ is finite;*

(ii) *D is acyclic and the partially ordered set of principal ideals of S does not contain infinite antichains.*

Proof. In the case where D is acyclic, our theorem follows from Lemma 57. If D has a cycle, then our theorem follows from Lemma 61.

The implication (iii)$\Rightarrow$(i) follows from Lemma 58. $\square$

5.4 Nil semigroups

The following lemma describes all divisibility D-saturated nilpotent semigroups.

Lemma 63 *Let D be a finite graph with edges, and let S be a nilpotent semigroup of degree n. Then S is divisibility D-saturated if any only if S is finite.*

Proof. The 'only if' part. Suppose to the contrary that S is infinite. The set $S \setminus S^2$ is infinite, as seen in Lemma 41, and induces an infinite null subgraph in $\mathrm{Div}(S)$. The graph D does not embed in this subgraph, a contradiction.

The 'if' part is vacuously true. □

The problem of describing all divisibility D-saturated nil semigroups is difficult. The following example shows that semigroups of this sort exist.

Example 64 Let $S = \{x \in I\!R : x \in (0,1)\} \cup \{\theta\}$, where multiplication is defined by the rule

$$xy = \begin{cases} x+y & \text{if } x, y, x+y \in (0,1), \\ \theta & \text{if } x, y \in (0,1) \text{ and } x+y \notin (0,1), \\ \theta & \text{if } x = \theta \text{ or } y = \theta. \end{cases}$$

Then S is nil, and the divisibility graph of S is a chain. That is, (x, y) is an edge if and only if $x > y$. Therefore S is divisibility D-saturated for every acyclic graph D, by Lemma 14.

Evidently, every nil semigroup that is divisibility D-saturated is not nilpotent. However, this condition is not sufficient for the 0-direct union of infinitely many copies of a nil semigroup is nil, but not divisibility D-saturated.

5.5 Subepigroups of completely 0-simple semigroups

Groups

It is not easy to characterise all divisibility D-saturated subsemigroups of groups, because divisibility D-saturated semigroups are not necessarily periodic. Thus a divisibility D-saturated subsemigroup of a group need not be a subgroup of that group, and the subgraph induced by the subsemigroup need not be isomorphic to K_∞, as the next two examples show.

Example 65 Let $S = (\mathbb{Q}^+, +)$, and let $M = (\mathbb{Q}, +)$. Then S is a subsemigroup of M, but not a subgroup of M. The divisibility graph induced by the elements of S is isomorphic to A_∞. Thus S is divisibility D-saturated, for all acyclic D.

Example 66 (Anderson, [3]) Let

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a, b \in I\!R, a, b > 0 \right\}.$$

Then S is a subsemigroup of $M_2(\mathbb{Q})$. Clearly, S is simple, and so $\mathrm{Div}(S) \cong K_\infty$. However, S is not a subgroup, since it has no idempotents.

In view of this, our attention is restricted to subepigroups of completely 0-simple semigroups.

Lemma 67 *A subepigroup S of a group G is a subgroup of G.*

Proof. Suppose that S is an epigroup, whose elements are contained in a group G, with identity e. By Lemma 42, $S = N/H$, where N is a nilextension and H is a group, also with identity e. Pick any element $n \in N$, if one exists. Then $ne = n \notin H$, and so H is not an ideal of S. This contradicts Lemma 42. Therefore the nilextension is empty, and so S is a subgroup of G. $\square$

Completely 0-simple semigroups

Recall that a completely 0-simple is isomorphic to a Rees matrix semigroup $M = \mathcal{M}^0(G; I, \Lambda; P)$. Throughout this section an arbitrary element of a maximal $\mathcal{H}^M$-class $G_{i\lambda}$ of M will be denoted by $g_{i\lambda}$, and if $G_{i\lambda}$ is a subgroup, then the identity of $G_{i\lambda}$ is denoted by $e_{g_{i\lambda}}$.

It is well known that a subsemigroup of an epigroup is not necessarily an epigroup, as evidenced in Example 65.

Lemma 68 *Let $M = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup. A subsemigroup S of M is an epigroup if and only if, for each maximal subgroup $G_{i\lambda}$ of M, $S \cap G_{i\lambda}$ is empty or a subgroup of S.*

Proof. The 'only if' part. Take any two elements $x, y \in S \cap G_{i\lambda}$, where $G_{i\lambda}$ is a maximal subgroup of M. The element y^m lies in a subgroup of S, for some positive integer m. Therefore $(y^m)^{-1} \in S$. We know that $(y^m)^{-1} = (y^{-1})^m$, by

[50], Lemma 1.1.5. Thus $y^{m-1}(y^m)^{-1} = y^{m-1}(y^{-1})^m = y^{-1}$, and so $y^{-1} \in S$. Therefore $xy^{-1} \in S$.

Obviously $xy^{-1} \in G_{i\lambda}$. Thus $xy^{-1} \in S \cap G_{i\lambda}$, and so $S \cap G_{i\lambda}$ is a subgroup of S, by Lemma 1.

The 'if' part. If $s \in S$ and s is not contained in a subgroup of S, then s is not contained in a subgroup of M, and so $s^2 = 0$, by [21], Proposition 3.2.7. Therefore S is an epigroup. □

Consider the following linear semigroup.

Example 69 Let M equal the Brandt semigroup of 3×3 monomial matrices with entries over $I\!R \setminus \{0\}$, and let S be the subsemigroup of M which is the union of

$$\left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

for all $x \in \mathbb{Q} \setminus \{0\}$ and $y \in I\!R \setminus \{0\}$. The elements of S that belong to subgroups of S are contained in the following sets:

$$G_x = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x \in \mathbb{Q} \right\} \text{ and } G_y = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid y \in I\!R \right\}.$$

Let γ be the equivalence relation on S with equivalence classes G_x and G_y, and let $\rho = \gamma^{\#}$. By Lemma 2, $|S/\rho| = 5$. However, the elements $\left\{ \begin{pmatrix} 0 & 0 & \sqrt{p} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \text{for } p \text{ prime} \right\}$ induce a null subgraph in $\mathrm{Div}(S)$, and so S is not D-saturated.

Example 69 shows that, in general $\mathcal{J}$ is not a congruence in linear semigroups, unlike the case for commutative semigroups, as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \pmod{\mathcal{J}},$$

but

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\not\equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \pmod{\mathcal{J}}.$$

Thus $\mathcal{J} \neq \mathcal{J}^{\#}$ in general for linear semigroups.

The next two results help to clarify the structure of subepigroups of completely 0-simple semigroups.

Lemma 70 *Let S be an epigroup such that $S \subseteq M = \mathcal{M}^0(G; I, \Lambda; P)$, and suppose that $g_{i_1\lambda_1}$ and $g_{i_2\lambda_2}$ belong to subgroups of S. Then $J^S_{g_{i_1\lambda_1}} \leq J^S_{g_{i_2\lambda_2}}$ implies that $J^S_{g_{i_1\lambda_1}} = J^S_{g_{i_2\lambda_2}}$.*

Proof. If $g_{i_1\lambda_1}$ and $g_{i_2\lambda_2}$ belong to the same subgroup in S, then the result if evident. Therefore we may assume that $g_{i_1\lambda_1}$ and $g_{i_2\lambda_2}$ belong to distinct maximal subgroups in S such that $J^S_{g_{i_1\lambda_1}} \leq J^S_{g_{i_2\lambda_2}}$. Hence there exist elements $a, b \in S^1$ such that $g_{i_1\lambda_1} = a g_{i_2\lambda_2} b$.

If $a = 1$, then $i_1 = i_2$ and $g_{i_1\lambda_1} \mathcal{R}^M g_{i_2\lambda_2}$. Then the product $g_{i_1\lambda_1} g_{i_2\lambda_2}$ lies in $G_{g_{i_2\lambda_2}}$, by Lemma 4. Therefore $g_{i_1\lambda_1} g_{i_2\lambda_2} (g_{i_1\lambda_1} g_{i_2\lambda_2})^{-1} g_{i_2\lambda_2} = g_{i_2\lambda_2}$, and so $J^S_{g_{i_2\lambda_2}} \leq J^S_{g_{i_1\lambda_1}}$.

The dual argument applies if $b = 1$.

If $a, b \neq 1$, then $(g_{i_2\lambda_2} b) g_{i_1\lambda_1} (a g_{i_2\lambda_2}) \in G_{g_{i_2\lambda_2}}$, by Lemma 4, and so

$$(g_{i_2\lambda_2} b g_{i_1\lambda_1} a g_{i_2\lambda_2})(g_{i_2\lambda_2} b g_{i_1\lambda_1} a g_{i_2\lambda_2})^{-1} g_{i_2\lambda_2} = g_{i_2\lambda_2}.$$

Thus $J^S_{g_{i_1\lambda_1}} \geq J^S_{g_{i_2\lambda_2}}$, and hence $J^S_{g_{i_1\lambda_1}} = J^S_{g_{i_2\lambda_2}}$ in this case, too. □

Lemma 71 *Let S be an epigroup such that $S \subseteq M = \mathcal{M}^0(G; I, \Lambda; P)$, and denote by G_S the set of all elements that belong to subgroups of S. For any* *fixed* $g_{i\lambda} \in G_S$ *let*

$$\begin{aligned} H &= \{g_{j\mu} \in G_S \mid g_{j\mu}\mathcal{J}^S g_{i\lambda}\}, \\ I_H &= \{j \in I \mid \exists g_{j\lambda} \in H\}, \\ \Lambda_H &= \{\mu \in \Lambda \mid \exists g_{i\mu} \in H\}. \end{aligned}$$

Then $C = \{g_{j\mu} \in S \mid j \in I_H, \mu \in \Lambda_H\} \cup \{0\}$ *is a completely* 0*-simple subsemigroup of* S.

Proof. Take any element $g_{i_1\lambda_1} \in C \setminus \{0\}$, not contained in H and where $i_1 \in I_H$, $\lambda_1 \in \Lambda_H$. By the definitions of I_H and Λ_H, there exist elements $g_{i_1\lambda_2}, g_{i_2\lambda_1} \in H$ such that $g_{i_1\lambda_2}\mathcal{R}^M g_{i_1\lambda_1}\mathcal{L}^M g_{i_2\lambda_1}$. Hence $e_{g_{i_1\lambda_2}} g_{i_1\lambda_1} = g_{i_1\lambda_1}$, by [21], Proposition 2.3.3. Thus $g_{i_1\lambda_2}(g_{i_1\lambda_2})^{-1}g_{i_1\lambda_1} = g_{i_1\lambda_1}$, and so $J^S_{g_{i_1\lambda_2}} \geq J^S_{g_{i_1\lambda_1}}$.

By Lemma 70, $J^S_{g_{i_1\lambda_2}} = J^S_{g_{i_2\lambda_1}}$, and so there exist elements $a, b \in S^1$ such that $g_{i_1\lambda_2} = a g_{i_2\lambda_1} b$. Hence $g_{i_2\lambda_1} b \neq 0$, and so $L^M_{g_{i_2\lambda_1}} \cap R^M_b$ is a subgroup of M, by Lemma 4. Since $L^M_b = L^M_{g_{i_1\lambda_2}}$, the element $g_{i_1\lambda_1} b$ is contained in $G_{i_1\lambda_2}$. Therefore $g_{i_1\lambda_1} b (g_{i_1\lambda_1} b)^{-1} g_{i_1\lambda_2} = g_{i_1\lambda_2}$, and so $J^S_{g_{i_1\lambda_1}} \geq J^S_{g_{i_1\lambda_2}}$.

Thus $J^S_{g_{i_1\lambda_2}} = J^S_{g_{i_1\lambda_1}}$. Therefore all elements of $C \setminus \{0\}$ are $\mathcal{J}^S$-equivalent. The product of any two elements of C^0 lies in C^0, by [21], Theorem 3.3.1. It follows that C^0 is a 0-simple subsemigroup of S. Since S is an epigroup, C^0 is completely 0-simple, by [21], Theorem 3.2.11. □

Lemma 72 *Let* S *be a subepigroup of* $M = \mathcal{M}^0(G; I, \Lambda; P)$, *where* G *is infinite, and let* D *be a finite graph. Let* E_S *be the set of all idempotents in* S,

$$\begin{aligned} G_I &= \{i \in I \mid G_{i*} \cap S \neq \emptyset, R^S_{g_{i\lambda}} \not\leq R^S_e, \forall g_{i\lambda} \in G_{i*} \cap S, e \in E_S\}, \\ G_\Lambda &= \{\lambda \in \Lambda \mid G_{*\lambda} \cap S \neq \emptyset, L^S_{g_{i\lambda}} \not\leq L^S_e, \forall g_{i\lambda} \in G_{*\lambda} \cap S, e \in E_S\}, \text{ and} \\ T &= \{g_{i\lambda} \in S \mid i \in G_I, \lambda \in G_\Lambda\}. \end{aligned}$$

Then the following statements are equivalent:

(i) S *is divisibility* D*-saturated;*

(ii) *for each subgroup* $G_{i\lambda}$ *of* M, $S \cap G_{i\lambda}$ *is empty or is a subgroup of* S, S *contains a finite number of maximal completely* 0*-simple semigroups,*

$|T|$, $|G_I|$ *and* $|G_\Lambda|$ *are finite, the union of all subgroups of* S *is infinite and, for each subset* $G_{i\lambda} \cap S$, *not contained in a completely* 0*-simple subsemigroup, a union of subgroups of* S *induces a congruence* $\rho_{i\lambda}$ *such that* $(G_{i\lambda} \cap S)/\rho_{i\lambda}$ *is finite;*

(iii) $S/\mathcal{J}^S$ *is finite.*

Proof. (i)$\Rightarrow$(ii): Suppose that $G_{i\lambda}$ is a subgroup of M. The fact that $S \cap G_{i\lambda}$ is empty or is a subgroup of S follows from Lemma 68.

Consider two distinct, maximal completely 0-simple subsemigroups, say C_j and C_k, where C_j, C_k, H_j and H_k, are defined in Lemma 71. Suppose that $(a, b) \in E(\operatorname{Div}(S))$, where $a \in V(C_j)$ and $b \in V(C_k)$. Pick two elements, $c \in H_j$ and $d \in H_k$. Then $(c, a), (b, d) \in E(\operatorname{Div}(S))$. Since the divisibility relation is transitive, $(c, d) \in E(\operatorname{Div}(S))$. Therefore $J_c^S \leq J_d^S$, and $J_c^S = J_d^S$, by Lemma 70. This implies that $H_j = H_k$, and that $C_j = C_k$, a contradiction. Therefore $(a, b) \notin E(\operatorname{Div}(S))$. Similarly, $(b, a) \notin E(\operatorname{Div}(S))$.

If S contains infinitely many maximal completely 0-simple subsemigroups, then the subset obtained by selecting one element from each maximal completely 0-simple subsemigroup induces a null subgraph in $\operatorname{Div}(S)$, which contradicts (i). Therefore S contains a finite number of maximal completely 0-simple subsemigroups. Denote them by $C_1, C_2, \ldots, C_m$, and let

$$\begin{aligned} I_{C_k} &= \{i \in I \mid \exists g_{i\lambda} \in C_k\}, \\ \Lambda_{C_k} &= \{\lambda \in \Lambda \mid \exists g_{i\lambda} \in C_k\}. \end{aligned}$$

Suppose that T is infinite. If T intersects finitely many $\mathcal{H}^M$-classes, then there exists an infinite subset U such that all elements of U belong to the same $\mathcal{H}^M$-class. Pick any two distinct elements $a, b \in U$. If $(a, b) \in E(\operatorname{Div}(S))$, then by Lemma 4, $a = xby$, where x, y belong to subgroups of S^1. If $x \neq 1$, then $a = e_x xby$, and so $R_a^S \leq R_{e_x}^S$, a contradiction. If $x = 1$, then $y \neq 1$, and so $L_a^S \leq L_{e_y}^S$, again a contradiction. Therefore $(a, b) \notin E(\operatorname{Div}(S))$. Similarly $(b, a) \notin E(\operatorname{Div}(S))$. Hence the set U induces an infinite null subgraph in $\operatorname{Div}(S)$, which contradicts (i).

Therefore T intersects infinitely many $\mathcal{H}^M$-classes of M. An infinite subset U of T may be selected such that no two elements of T belong to the same $\mathcal{H}^M$-class. Then Corollary 21 states that U contains an infinite countable subset V such that the subgraph H of $\operatorname{Div}(S)$ with all elements of V is null, or isomorphic to A_∞, D_∞ or K_∞.

The first case contradicts (i), and so is impossible.

If $H \cong D_\infty$ or $H \cong K_\infty$, and V contains an infinite subset W such that no two elements of W are $\mathcal{L}^M$-equivalent, then the elements of W can be indexed by the positive integers such that

$$u_i w_i x_i = w_{i+1}, \tag{5.1}$$

for $x_i \in S, u_i \in S^1$ and $1 \leq i$. Therefore $x_i \mathcal{L}^M w_{i+1}$, and so $L^M_{x_i} \neq L^M_{x_j}$ for $i \neq j$.

We know by (5.1) that $u_{i+1}u_i w_i x_i x_{i+1} = w_{i+2}$. Therefore $x_i x_{i+1} \neq 0$, and so $R^M_{x_{i+1}} \cap L^M_{x_i}$ is a subgroup of M, for all $1 \leq i$. If $x_i \mathcal{R}^M x_j$, for some $i < j$, then $R^M_{x_j} \cap L^M_{x_{j-1}} = R^M_{x_i} \cap L^M_{x_{j-1}}$ is a subgroup of M. The element $x_i x_{i+1} \dots x_{j-1} \neq 0$, by (5.1), and lies in $R^M_{x_i} \cap L^M_{x_{j-1}}$. Hence $w_j = w_j e_{x_i x_{i+1} \dots x_{j-1}}$, and so $L^S_{w_j} \leq L^S_{e_{x_i \dots x_{j-1}}}$, which contradicts the definition of T. Therefore $R^M_{x_i} \neq R^M_{x_j}$, for $i \neq j$.

Consider the subgraph induced by the elements $x_i \in S$. By Corollary 21, there exists a countable subset $X = x_{i_1}, x_{i_2}, \dots$ of $x_1, x_2, \dots$ such that the subgraph F of $\mathrm{Div}(S)$ induced by the elements of X is null, or isomorphic to A_∞, D_∞ or K_∞.

If F is null, we get a contradiction with (i), and so this case is impossible.

If $F \cong D_\infty$ or $F \cong K_\infty$, then the elements of X can be indexed by the positive integers such that $J^S_{x_{i_k}} > J^S_{x_{i_\ell}}$, for $1 \leq k < \ell$. Then

$$a_{i_k} x_{i_k} b_{i_k} = x_{i_{k+1}}, \tag{5.2}$$

for $a_{i_k}, b_{i_k} \in S$ and $1 \leq k$.

Now $u_{i_2-1}u_{i_2-2} \dots u_{i_1} w_{i_1} x_{i_1} x_{i_1+1} \dots x_{i_2-1} = w_{i_2}$, by (5.1). Thus the element $x_{i_1} \dots x_{i_2-1} \neq 0$, and lies in $R^M_{x_{i_1}} \cap L^M_{x_{i_2-1}}$, by Lemma 4.

The element $a_{i_1} x_{i_1} \neq 0$, by (5.2), and so $L^M_{a_{i_1}} \cap R^M_{x_{i_1}}$ is a subgroup of M, by Lemma 4. Clearly $R^M_{x_{i_1}} = R^M_{x_{i_1} x_{i_1+1} \dots x_{i_2-1}}$, since $x_{i_1} x_{i_1+1} \dots x_{i_2-1} \neq 0$. Therefore $a_{i_1} x_{i_1} x_{i_1+1} \dots x_{i_2-1} \neq 0$, as $L^M_{a_{i_1}} \cap R^M_{x_{i_1} x_{i_1+1} \dots x_{i_2-1}} = L^M_{a_{i_1}} \cap R^M_{x_{i_1}}$ is a subgroup. It follows from (5.2) that $R^M_{a_{i_1}} = R^M_{x_{i_2}}$. Thus the element $a_{i_1} x_{i_1} x_{i_1+1} \dots x_{i_2-1}$ lies in $R^M_{a_{i_1}} \cap L^M_{x_{i_2-1}} = R^M_{x_{i_2}} \cap L^M_{x_{i_2-1}}$, also a subgroup.

Since $L^M_{x_{i_2-1}} = L^M_{w_{i_2}}$, we have $w_{i_2} = w_{i_2} e_{a_{i_1} x_{i_1} \dots x_{i_2-1}}$, by [21], Proposition 2.3.3. Thus $L^S_{w_{i_2}} \leq L^S_{e_{a_{i_1} x_{i_1} \dots x_{i_2-1}}}$. This contradicts the definition of T and shows that $F \not\cong D_\infty$ and $F \not\cong K_\infty$.

If $F \cong A_\infty$, then the elements of X can be indexed by the positive integers such that $J^S_{x_{i_k}} < J^S_{x_{i_\ell}}$, for $1 \leq k < \ell$. Then $x_{i_k} = a_{i_k} x_{i_{k+1}} b_{i_k}$, for $a_{i_k}, b_{i_k} \in S$ and $1 \leq k$. The dual argument yields that $L^S_{w_{i_2+1}} \leq L^S_{e_{b_{i_1}} x_{i_1+1} \cdots x_{i_2}}$, a contradiction again. Therefore $F \not\cong A_\infty$.

Thus we have shown that, if $H \cong D_\infty$ or $H \cong K_\infty$, then V does not contain an infinite subset W such that no two elements of W are $\mathcal{L}^M$-equivalent.

If $H \cong D_\infty$ or $H \cong K_\infty$, and V contains an infinite subset W such that no two elements of W are $\mathcal{R}^M$-equivalent, then the elements of W can be indexed by the positive integers such that $u_i w_i x_i = w_{i+1}$, for $x_i \in S^1, u_i \in S$ and $1 \leq i$. The dual argument implies that the set $u_i \in S$ contains an infinite subset X such that the subgraph F of $\mathrm{Div}(S)$ induced by the elements of X is not null, or not isomorphic to A_∞, D_∞ or K_∞, a contradiction.

Thus $H \not\cong D_\infty$ and $H \not\cong K_\infty$. Similarly, it can be shown that $H \not\cong A_\infty$. This contradicts Corollary 21. Therefore T is finite.

Suppose to the contrary that G_I is infinite. Since T is finite, there exists an infinite set

$$W = \{g_{i\lambda} \in S \mid R^M_{g_{i\lambda}} \neq R^M_{g_{j\mu}} \text{ for } i \neq j, \text{ where } i, j \in G_I, \lambda, \mu \in \Lambda \setminus G_\Lambda, \}.$$

By Corollary 21, the subgraph of $\mathrm{Div}(S)$ with all vertices being in W contains an infinite null subgraph, or a subgraph G isomorphic to A_∞, D_∞ or K_∞. The first case is impossible by (i).

Suppose that $G \cong A_\infty$ or $G \cong K_\infty$. Index a countable subset of the vertices of G by the positive integers such that $(g_{i_j\lambda_j}, g_{i_\ell\lambda_\ell}) \in E(G)$, for $1 \leq j < \ell$. Clearly, there exist $a_j \in S, b_j \in S^1$ such that $g_{i_j\lambda_j} = a_j g_{i_{j+1}\lambda_{j+1}} b_j$, for $1 \leq j$.

For $1 \leq k \leq m$, put

$$V_k = \{g_{i\lambda} \in S \mid i \in G_I, \lambda \in \Lambda_{C_k}\}.$$

Since the number of maximal completely 0-simple semigroups in S is finite, there exist two elements $a_p, a_q \in V_k$, where $p < q$. It is easily seen that $g_{i_p\lambda_p} = a_p a_{p+1} g_{i_{p+2}\lambda_{p+2}} b_{p+1} b_p$. Therefore $a_p a_{p+1} \neq 0$, and hence $L^M_{a_p} \cap R^M_{a_{p+1}}$ is a subgroup of M, by Lemma 4. Similarly, the product $a_{p+1} a_{p+2} \cdots a_q$ is nonzero, and lies in $R^M_{a_p+1} \cap L^M_{a_q} \in V_k$.

There exists $h \in I_{C_k}$ such that $R^M_h \cap L^M_{a_q} \cap S$ is a maximal subgroup in C_k. Pick any $g \in R^M_h \cap L^M_{a_p} \cap S$. The element $a_{p+1} a_{p+2} \cdots a_q g$ is nonzero, by

Lemma 4, and is contained in $R^M_{a_{p+1}} \cap L^M_g = R^M_{a_{p+1}} \cap L^M_{a_p}$, a subgroup of M. Therefore $e_{a_{p+1}a_{p+2}\dots a_q g}\mathcal{R}^M g_{i_{p+1}\lambda_{p+1}}$, and by [21], Proposition 2.3.3 we get $g_{i_{p+1}\lambda_{p+1}} = e_{a_{p+1}a_{p+2}\dots a_q g}\, g_{i_{p+1}\lambda_{p+1}}$. Hence $R^S_{e_{a_{p+1}a_{p+2}\dots a_q g}} \geq R^S_{g_{i_{p+1}\lambda_{p+1}}}$, contradicting the definition of G_I. Thus $G \ncong A_\infty$ and $G \ncong K_\infty$.

Suppose that $G \cong D_\infty$. Then a countable subset of the vertices of G may be indexed by the positive integers such that $(g_{i_j\lambda_j}, g_{i_\ell\lambda_\ell}) \in E(G)$, for $1 \leq \ell < j$. There exist elements $a_j \in S, b_j \in S^1$ such that $a_j g_{i_j\lambda_j} b_j = g_{i_{j+1}\lambda_{j+1}}$, for $1 \leq j$. In this case $a_{p+1}a_p \neq 0$, for all $p \in \mathbb{Z}^+$, and so $L^M_{a_{p+1}} \cap R^M_{a_p}$ is a subgroup of M.

The same reasoning as in the preceding case leads to a contradiction again in this case. Since k is finite, $a_p, a_q \in V_k$, for some $p < q$ and $1 \leq k \leq m$. There exists $h \in I_{C_k}$ such that $R^M_h \cap L^M_{a_p} \cap S$ is a maximal subgroup in C_k. For any $g \in R^M_h \cap L^M_{a_q} \cap S$ it follows that $R^S_{e_{a_{q-1}\dots a_p g}} \geq R^S_{g_{i_q\lambda_q}}$. This contradiction shows that $G \ncong D_\infty$.

Therefore we conclude that G_I is finite. A similar argument shows that G_Λ is finite.

Suppose that the set G_S of all elements of S contained in subgroups of S is finite. Then C_k is finite, for all $1 \leq k \leq m$. Consequently, $I \setminus G_I$ and $\Lambda \setminus G_\Lambda$ are finite. Since G_I and G_Λ are finite, I and Λ are finite too. Therefore $(S \setminus G_S) \cap G_{i\lambda}$ is infinite, for some $i \in I, \lambda \in \Lambda$. Denote by H the subgraph of $\mathrm{Div}(S)$ induced by the elements of $(S \setminus G_S) \cap G_{i\lambda}$. If $(a, b) \in E(H)$, then $a = xby$, where x, y belong to subgroups in S^1. Therefore every vertex in H has finite indegree. This implies that H does not contain a subgraph isomorphic to A_∞, D_∞ or K_∞. Then Corollary 21 tells us that there exists an infinite subset of $(S \setminus G_S) \cap G_{i\lambda}$ that induces a null subgraph, contradicting (i). Thus the union of all subgroups of S is infinite.

Consider an infinite subset $G_{i\lambda} \cap S$ not contained in a completely 0-simple semigroup C_k. Since T is finite, either $i \in I \setminus G_I$ or $\lambda \in \Lambda \setminus G_\Lambda$. Let $\gamma_{i\lambda}$ be the equivalence relation on S whose equivalence classes are the subgroups $G_{i\mu} \cap S$ and $G_{j\lambda} \cap S$ of S, for all $j \in I, \lambda \in \Lambda$. Put $\rho_{i\lambda} = (\gamma_{i\lambda})^\#$.

Suppose that $(G_{i\lambda} \cap S)/\rho_{i\lambda}$ is infinite. Then there exists an infinite subset U of $G_{i\lambda} \cap S$ such that no two elements belong to the same $\rho_{i\lambda}$-class.

Take any two elements $a, b \in U$. If $i \in I \setminus G_I$, $\lambda \in \Lambda \setminus G_\Lambda$ and (a, b) is an edge of $\mathrm{Div}(S)$, then $a = xby$, where x and y belong to subgroups $G_{i\mu} \cap S$ and $G_{j\lambda} \cap S$, respectively. Then $(x, e_x) \in \rho_{i\lambda}$ and $(y, e_y) \in \rho_{i\lambda}$. This implies that $(xby, e_xbe_y) = (a, b) \in \rho_{i\lambda}$, which contradicts our assumption. The same argument shows that the edge $(b, a) \notin E(\mathrm{Div}(S))$.

If $(a, b) \in E(\mathrm{Div}(S))$ for $a, b \in U$, $i \in I \setminus G_I$ and $\lambda \in G_\Lambda$, then $a = xb$, where x belongs to a subgroup $G_{i\mu} \cap S$. Then $(x, e_x) \in \rho_{i\lambda}$, and so $(xb, e_xb) = (a, b) \in \rho_{i\lambda}$, which contradicts the definition of U. Similarly, $(b, a) \notin E(\mathrm{Div}(S))$.

If $(a, b) \in E(\mathrm{Div}(S))$ for $a, b \in U$, $i \in G_I$ and $\lambda \in \Lambda \setminus G_\Lambda$, then the dual argument leads to a contradiction again. Similarly, (b, a) is not an edge of $\mathrm{Div}(S)$.

In all cases the elements of U induce a null subgraph in $\mathrm{Div}(S)$, which contradicts (i). Hence $(G_{i\lambda} \cap S)/\rho_{i\lambda}$ is finite.

(ii)$\Rightarrow$(iii): Consider an infinite subset $G_{i\lambda} \cap S$ not contained in C_k, for $1 \leq k \leq m$. The equivalence relation $\gamma_{i\lambda}$ is contained in $\mathcal{J}^S$, and therefore $\rho_{i\lambda} \subseteq (\mathcal{J}^S)^\#$. Example 69 shows that $(\mathcal{J}^S)^\# \neq \mathcal{J}^S$ in general, and so it is not immediate that $\rho_{i\lambda} \subseteq \mathcal{J}^S$, when restricted to $G_{i\lambda} \cap S$.

Pick two elements $a, b \in G_{i\lambda} \cap S$ such that $(a, b) \in \rho_{i\lambda}$. By Lemma 2, there exists a sequence

$$a = z_1 \to z_2 \to \ldots \to z_n = b$$

of elementary $\gamma_{i\lambda}$-transitions connecting a to b.

If $i \in I \setminus G_I, \lambda \in \Lambda \setminus G_\Lambda$, then there exist subgroups $G_{i\mu} \cap S, G_{j\lambda} \cap S$ and elements $x, y \in S^1$, $c, d \in (G_{i\mu} \cup G_{j\lambda}) \cap S)$ such that $z_1 = xcy, z_2 = xdy$. By the definition of $\gamma_{i\lambda}$, either $c, d \in G_{i\mu} \cap S$ or $c, d \in G_{j\lambda} \cap S$, for some $\mu \in \Lambda \setminus G_\Lambda$ or $j \in I \setminus G_I$.

If $c, d \in G_{i\mu} \cap S$, for some $\mu \in \Lambda \setminus G_\Lambda$, then $xc, xd \in G_{i\mu} \cap S$. Hence $z_1 = xc(xd)^{-1}z_2$ and $z_2 = xd(xc)^{-1}z_1$. If $c, d \in G_{j\lambda} \cap S$, for some $j \in I \setminus G_I$, then $cy, dy \in G_{j\lambda} \cap S$. Therefore $z_1 = z_2(dy)^{-1}cy$, $z_2 = z_1(cy)^{-1}dy$. In both cases $z_1 \mathcal{J}^S z_2$. Induction on z_k shows that $a \mathcal{J}^S b$, and so $\rho_{i\lambda} \subseteq \mathcal{J}^S$, when restricted to $G_{i\lambda} \cap S$.

If $i \in I \setminus G_I, \lambda \in G_\Lambda$ (resp., $i \in G_I, \lambda \in \Lambda \setminus G_\Lambda$), then there exist subgroups $G_{i\mu} \cap S$ (resp., $G_{j\lambda} \cap S$), and elements $x, y \in S^1$, $c, d \in G_{i\mu} \cap S$ (resp., $c, d \in G_{j\lambda} \cap S$) such that $z_1 = xcy, z_2 = xdy$. The same argument shows that $\rho_{i\lambda} \subseteq \mathcal{J}^S$, when restricted to $G_{i\lambda} \cap S$.

Thus

$$|(G_{i\lambda}\cap S)/\mathcal{J}^S| \le |(G_{i\lambda}\cap S)/\rho_{i\lambda}|, \tag{5.3}$$

and so this set is finite.

Next, we show that for any fixed $i \in G_I$, $\lambda \in \Lambda_{C_k}$,

$$|\cup_{\lambda'\in\Lambda_{C_k}} (G_{i\lambda'}\cap S)/\mathcal{J}^S| = |(G_{i\lambda}\cap S)/\mathcal{J}^S|.$$

Pick any $|(G_{i\lambda}\cap S)/\mathcal{J}^S|+1$ elements, say $g_{i\lambda_j}$, for $1 \le j \le |(G_{i\lambda}\cap S)/\mathcal{J}^S|+1$ from the subset $\cup_{\lambda'\in\Lambda_{C_k}}(G_{i\lambda'}\cap S)/\mathcal{J}^S$. Denote this set by H. For each element $g_{i\lambda_j} \in H$, there exists $i_j \in I_{C_k}$ such that $G_{i_j\lambda_j}$ is a subgroup. Take any element $g_{i_j\lambda} \in G_{i_j\lambda}\cap S$. The map

$$\phi_{g_{i_j\lambda}} : G_{i\lambda_j} \to G_{i\lambda},$$

defined by $\phi_{g_{i_j\lambda}}(g_{i\lambda_j}) = g_{i\lambda_j}g_{i_j\lambda}$ is a bijection (see [21], §2.2).

There exists $p \in I_{C_k}$ such that $G_{p\lambda}$ is a maximal subgroup. Therefore each element $s \in G_{i_j\lambda}\cap S$ has an inverse $s' \in G_{p\lambda_j}\cap S$, by [21], Theorem 2.3.4. For each element $g_{i_j\lambda}$, let s_j be the inverse of $g_{i_j\lambda}$ contained in $G_{p\lambda_j}\cap S$. The map

$$\phi_{s_j} : G_{i\lambda} \to G_{i\lambda_j},$$

defined by $\phi_{s_j}(g_{i\lambda}) = g_{i\lambda}s_j$ is also a bijection. Indeed $\phi_{g_{i_j\lambda}}$ and ϕ_{s_j} are mutually inverse bijections. Therefore

$$J^S_{g_{i\lambda_j}} \ge J^S_{\phi_{g_{i_j\lambda}}(g_{i\lambda_j})} \ge J^S_{\phi_{s_j}(\phi_{g_{i_j\lambda}}(g_{i\lambda_j}))} = J^S_{g_{i\lambda_j}}. \tag{5.4}$$

The elements $\phi_{g_{i_j\lambda}}(g_{i\lambda_j}) \in G_{i\lambda}\cap S$, for all $1 \le j \le |(G_{i\lambda}\cap S)/\mathcal{J}^S|+1$. Hence at least two elements, say $\phi_{g_{i_p\lambda}}(g_{i\lambda_p})$ and $\phi_{g_{i_q\lambda}}(g_{i\lambda_q})$ are $\mathcal{J}^S$-equivalent. By (5.4),

$$J^S_{g_{i\lambda_p}} = J^S_{\phi_{g_{i_p\lambda}}(g_{i\lambda_p})} = J^S_{\phi_{g_{i_q\lambda}}(g_{i\lambda_q})} = J^S_{g_{i\lambda_q}}.$$

Therefore $|\cup_{\lambda'\in\Lambda_{C_k}} (G_{i\lambda'}\cap S)/\mathcal{J}^S| < |(G_{i\lambda}\cap S)/\mathcal{J}^S|+1$, and so

$$|\cup_{\lambda'\in\Lambda_{C_k}} (G_{i\lambda'}\cap S)/\mathcal{J}^S| = |(G_{i\lambda}\cap S)/\mathcal{J}^S|,$$

for all $\lambda \in \Lambda_{C_k}$ and $i \in G_I$.

The dual argument will show, for any fixed $\lambda \in G_\Lambda$ and $i \in I_{C_k}$, that

$$|\cup_{i'\in I_{C_k}} (G_{i'\lambda}\cap S)/\mathcal{J}^S| = |(G_{i\lambda}\cap S)/\mathcal{J}^S|,$$

and, for any fixed $\lambda \in \Lambda \setminus G_\Lambda$, $i \in I \setminus G_I$ and $j \neq k$, that

$$|\cup_{i' \in I_{C_j}, \lambda' \in \Lambda_{C_k}} (G_{i'\lambda'} \cap S)/\mathcal{J}^S| = |(G_{i\lambda} \cap S)/\mathcal{J}^S|.$$

Evidently,

$$|\cup_{i' \in I_{C_k}, \lambda' \in \Lambda_{C_k}} (G_{i'\lambda'} \cap S)/\mathcal{J}^S| = |C_k/\mathcal{J}^S| = 1.$$

Let

$$\begin{aligned}
N_{k,\lambda} &= |\cup_{i \in I_{C_k}} (G_{i\lambda} \cap S)/\mathcal{J}^S|, \text{ for } \lambda \in G_\Lambda, \\
P_{k,i} &= |\cup_{\lambda \in \Lambda_{C_k}} (G_{i\lambda} \cap S)/\mathcal{J}^S|, \text{ for } i \in G_I, \text{ and} \\
Q_{k,j} &= \begin{cases} |\cup_{i \in I_{C_j}, \lambda \in \Lambda_{C_k}} (G_{i\lambda} \cap S)/\mathcal{J}^S| & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}
\end{aligned}$$

Hence

$$\begin{aligned}
|S/\mathcal{J}^S| &\leq \sum_{k=1}^{m} \Bigg(\sum_{\lambda \in G_\Lambda} |\cup_{i \in I_{C_k}} (G_{i\lambda} \cap S)/\mathcal{J}^S| + \sum_{i \in G_I} |\cup_{\lambda \in \Lambda_{C_k}} (G_{i\lambda} \cap S)/\mathcal{J}^S| \\
&\quad + \sum_{j=1}^{m} |\cup_{i \in I_{C_k}, \lambda \in \Lambda_{C_j}} (G_{i\lambda} \cap S)/\mathcal{J}^S| \Bigg) + \sum_{i \in G_I, \lambda \in G_\Lambda} |(G_{i\lambda} \cap S)/\mathcal{J}^S| \\
&= \sum_{k=1}^{m} \left(\sum_{\lambda \in G_\Lambda} N_{k,\lambda} + \sum_{i \in G_I} P_{k,i} + \sum_{j=1}^{m} Q_{k,j} \right) + |T|,
\end{aligned}$$

which is finite.

The implication (iii)$\Rightarrow$(i) follows from Lemma 58. $\square$

5.6 Monomial matrix semigroups

Throughout this section an arbitrary $\mathcal{H}$-class of the Rees quotient semigroup $M_j/M_{j-1} = \mathcal{M}^0(G_j; I_j, \Lambda_j; P_j)$ is denoted by $G^j_{i\lambda}$, and an arbitrary element of $G^j_{i\lambda}$ is denoted by $g^j_{i\lambda}$. The $\mathcal{L}$-class (resp., $\mathcal{R}$-class) containing $G^j_{i\lambda}$ is denoted by $G^j_{*\lambda}$ (resp., G^j_{i*}).

Recall from Chapter 2 that a semisimple semigroup admits a principal series, where each factor is either simple or 0-simple. For a semisimple epigroup with a finite ideal series

$$\{0\} = M_0 \subset M_1 \subset \ldots \subset M_n = M_n(G),$$

each factor M_j/M_{j-1} is either completely simple or completely 0-simple, by [52], Theorem 1.1.

The next lemma characterises subepigroups of semisimple epigroups.

Lemma 73 *A subsemigroup S of a semisimple epigroup M is an epigroup if and only if, for each maximal subgroup G of M, $G \cap S$ is empty or a subgroup of S.*

Proof. The 'only if' part follows as in Lemma 68.

The 'if' part. Suppose that M has a finite ideal series of length n. Take any element $s \in M_j/M_{j-1} \cap S$. Either s^2 lies in a subgroup of $M_j/M_{j-1} \cap S$ or $s^2 \in M_{j-1} \cap S$, by [21], Lemma 3.2.7. Easy induction on j shows that either $s^{2^j} = 0$, or s^{2^j} lies in a subgroup of S. Hence S is an epigroup. □

Consider an epigroup M with a finite ideal series of length n, and let S be a subepigroup of M. Evidently, if $((M_j/M_{j-1}) \cap S)/\mathcal{J}^{(M_j/M_{j-1})\cap S}$ is finite for $1 \leq j \leq n$, then $S/\mathcal{J}^S$ is finite and S is D-saturated. The following example demonstrates that the converse is not true.

Example 74 Let S be the subepigroup of $M_2(\mathbb{Q})$ such that $S = A \cup B \cup \theta$, where

$$A = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in \mathbb{Q} \right\}, B = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \mid z \in \mathbb{Q} \right\}, \theta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The subepigroup S has a finite ideal series, namely

$$\theta \subset B \cup \theta \subset S,$$

and so is D-saturated. The factor $(B \cup \theta)/\theta$ has an infinite null divisibility graph, and so $(B \cup \theta)/\theta$ is not D-saturated.

The following lemma deals with the case where a principal factor of a semigroup is finite.

Lemma 75 *Let E be an infinite semigroup which is an ideal extension of S by Q, where Q is finite, and let D be a finite graph with at least one edge. Then E is D-saturated if and only if S is D-saturated.*

Proof. The 'only if' part. Suppose that D contains a cycle. Then E contains an infinite subset T such that $\mathrm{Div}(T)^E \cong K_\infty$, by Corollary 21. Hence the subgraph $\mathrm{Div}(T \cap S)^E \cong K_\infty$.

If $D_\infty \subseteq \mathrm{Div}(T \cap S)^S$ or $\mathrm{Div}(T \cap S)^S$ contains an infinite null subgraph, then there exists a countable subset $t_1, t_2, \ldots$ of $T \cap S$ such that (t_i, t_j) is an edge in $\mathrm{Div}(T \cap S)^E$, but $(t_i, t_j) \notin E(\mathrm{Div}(T \cap S)^S)$, for $1 \leq i < j$. Therefore $t_i = a_{i,j} t_j b_{j,i}$, where either $a_{i,j} \in Q$ or $b_{j,i} \in Q$, for all $1 \leq i < j$. We label the edges $(t_i, t_j) \in E(\mathrm{Div}(T \cap S)^E)$, where $1 \leq i < j$ according to the rule

$$(t_i, t_j) = \begin{cases} \mathrm{q} & \text{if } a_{i,j} = q \in Q \text{ and } b_{i,j} \in S^1, \\ \mathrm{r} & \text{if } a_{i,j} \in S^1 \text{ and } b_{j,i} = r \in Q, \\ \mathrm{z} & \text{if } a_{i,j}, b_{j,i} \in Q^1. \end{cases} \tag{5.5}$$

Lemma 20 tells us that there exists an infinite subset $V = v_1, v_2, \ldots$ of $T \cap S$ such that all two-element subsets of V have the same label. We may assume that the elements of V are indexed so that $(v_i, v_j) \notin E(\mathrm{Div}(V)^S)$, for all $1 \leq i < j$.

Since the semigroup Q/S is finite it is periodic, and so for each label $q, r \in Q$ there exist positive integers n_q and n_r, respectively, such that q^{n_q} and r^{n_r} are idempotent.

If the two-element subsets of V are labelled by q, then $v_i = q v_{i+1} c_{i+1,i}$, for some $q \in Q$, $c_{i+1,i} \in S$ and all $i \geq 1$. Then

$$v_1 = q^{n_q} v_{n_q+1} c_{n_q+1,n_q} c_{n_q,n_q-1} \cdots c_{2,1}, \tag{5.6}$$

$$v_{n_q+1} = q^{n_q} v_{2n_q+1} c_{2n_q+1,2n_q} c_{2n_q,2n_q-1} \cdots c_{n_q+2,n_q+1}. \tag{5.7}$$

Equation (5.7) implies that $v_{n_q+1} = q^{n_q} v_{n_q+1}$. Substituting into equation (5.6) gives $v_1 = v_{n_q+1} c_{n_q+1,n_q} c_{n_q,n_q-1} \cdots c_{2,1}$. This means that (v_1, v_{n_q+1}) is an edge in $E(\mathrm{Div}(V)^S)$, a contradiction.

If the two-element subsets of V are labelled by r, then $v_i = d_{i,i+1} v_{i+1} r$, for some $d_{i+1,i} \in S, r \in Q$. The dual argument shows that (v_1, v_{n_r+1}) is an edge in $E(\mathrm{Div}(V)^S)$, again a contradiction.

The third possibility implies that the outdegree of every vertex in $\mathrm{Div}(V)^E$ is finite, contradicting the fact that $\mathrm{Div}(V)^E \cong K_\infty$.

Thus $D_\infty \not\subseteq \mathrm{Div}(T \cap S)^S$ and $\mathrm{Div}(T \cap S)^S$ does not contain an infinite null subgraph. Similarly, $A_\infty \not\subseteq \mathrm{Div}(T \cap S)^S$. Therefore $K_\infty \subseteq \mathrm{Div}(T \cap S)^S$, by Corollary 21. The graph D embeds in this subgraph, and so S is D-saturated.

If D is acyclic, then D embeds in A_∞, D_∞ and K_∞, by Lemma 14. It suffices to show that S does not contain an infinite set U such that $\mathrm{Div}(U)^S$ is null.

Suppose to the contrary that there exists an infinite subset U of S such that $\mathrm{Div}(U)^S$ is null.

The preceding argument implies that $K_\infty \not\subseteq \mathrm{Div}(U)^E$.

If $A_\infty \subseteq \mathrm{Div}(U)^E$, then there exists a countable subset $u_1, u_2, \ldots$ of U such that $(u_i, u_j) \in E(\mathrm{Div}(U)^E)$, but $(u_i, u_j) \notin E(\mathrm{Div}(U)^S)$, for $1 \leq i < j$. This is the same condition as in the previous case, where E contains an infinite subset T such that $K_\infty \cong \mathrm{Div}(T \cap S)^E$ and $\mathrm{Div}(T \cap S)^S$ is null or contains D_∞. If the edges of $\mathrm{Div}(U)^E$ are labelled according to (5.5), then U contains an infinite subset V such that either $(v_1, v_{1+n_q}) \in E(\mathrm{Div}(V)^S)$, or (v_1, v_{1+n_r}) is and edge in $\mathrm{Div}(V)^S$, or the outdegree of every vertex in $\mathrm{Div}(V)^E$ is finite. This contradiction implies that $A_\infty \not\subseteq \mathrm{Div}(U)^E$.

Similarly, $D_\infty \not\subseteq \mathrm{Div}(U)^E$.

Corollary 21 implies that $\mathrm{Div}(U)^E$ contains an infinite null subgraph. This contradicts the hypothesis that E is D-saturated. Thus we conclude that there does not exist an infinite subset U of S such that $\mathrm{Div}(U)^S$ is null.

Corollary 21 now shows that $\mathrm{Div}(U)^S$ contains an infinite subgraph G isomorphic to A_∞, D_∞ or K_∞. The graph D embeds in G, by Lemma 14, and hence in $\mathrm{Div}(U)^S$. Thus S is D-saturated.

The 'if' part. Suppose that S is D-saturated. Then every infinite subset T of E contains an infinite subset $U = T \cap S$ of S. The graph D embeds in the subgraph induced by the elements of U, and hence E is D-saturated. □

The following version of Ramsey's theorem ([47], Theorem B) together with Lemma 77 is needed for the proofs of Lemma 78 and Theorem 79.

Theorem 76 ([39], Theorem 4.1.3) *Let r, k, n be positive integers with $1 \leq r \leq k$, and let $\mathcal{P}_r(X) = \{Y \subseteq X \mid |Y| = r\}$, the set of all r-subsets of a set X. There exists an integer $R(r,k,n)$ such that for each set X of cardinality $R(r,k,n)$ and each partition $Y_1, Y_2, \ldots Y_n$ of $\mathcal{P}_r(X)$ in n blocks, there exists a k-subset Y of X and a block Y_i such that $\mathcal{P}_r(Y) \subseteq Y_i$.*

Lemma 77 ([39], Theorem 4.14) *Let $\phi : A^+ \to N$ be a mapping from A^+ to a set N with cardinality n. For each $k \geq 1$, each word $w \in A^+$ of length $R(2, k+1, n)$ contains a factor $w_1 w_2 \ldots w_k$ with $w_i \in A^+$ and*

$$\phi(w_i \ldots w_{i'}) = \phi(w_j \ldots w_{j'}),$$

for all pairs $(i, i'), (j, j')(1 \leq i \leq i' \leq k$ and $1 \leq j \leq j' \leq k)$.

Lemma 78 *Let S be a subepeigroup of the monomial matrix semigroup $M_n(G)$, and denote by G_S the set of elements of S that belong to subgroups of S. Let D be a finite graph with at least one edge. If S is D-saturated, then every subsemigroup of S containing G_S is D-saturated.*

Proof. Let U be a subsemigroup of S containing G_S and take any infinite subset T of U. Then T contains an infinite countable subset $V = v_1, v_2, \ldots$ such that $\mathrm{Div}(V)^S$ is null, or isomorphic to A_∞, D_∞ or K_∞, by Corollary 21.

The first case contradicts the fact that S is D-saturated, and so is impossible.

Suppose that $\mathrm{Div}(V)^S \cong A_\infty$ or $\mathrm{Div}(V)^S \cong K_\infty$ and that D embeds in $\mathrm{Div}(V)^S$. Then there exist elements $a_m, b_m \in S^1$ such that $v_m = a_m v_{m+1} b_m$, for $m \geq 1$. Therefore the infinite product $a_1 a_2 \ldots$ is nonzero.

Consider the free semigroup A^+, where $A = S^1$, and define a map $\phi : A^+ \to N$, by $\phi(g^j_{i\lambda}) = (j, i, \lambda)$ and $\phi(1) = 1$. Since $M_n(G)$ has a finite ideal series and I_j, Λ_j are finite, for all $1 \leq j \leq n$, N is finite. By Lemma 77, $a_1 a_2 \ldots a_{R(2,R(2,|D|+1,|N|)+1,|N|)}$ contains a factor

$$w_1 w_2 \ldots w_{R(2,|D|+1,|N|)},$$

where $\phi(w_p) = \phi(w_q) = \phi(w_p w_q)$, for all $1 \leq p \leq q \leq R(2, |D|+1, |N|)$. For $1 \leq m \leq R(2, |D|+1, |N|)$, let

$$\begin{aligned} w_m &= a_{j_m} a_{j_m+1} \ldots a_{j_{(m+1)}-1}, \\ x_m &= b_{j_{(m+1)}-1} b_{j_{(m+1)}-2} \ldots b_{j_m}. \end{aligned}$$

By Lemma 77, $x_{R(2,|D|+1,|N|)} x_{R(2,|D|+1,|N|)-1} \ldots x_1$ contains a factor

$$z_{|D|} z_{|D|-1} \ldots z_1,$$

where $\phi(z_q) = \phi(z_p) = \phi(z_q z_p)$, for all $1 \leq p \leq q \leq |D|$. Let

$$\begin{aligned} z_m &= x_{t_{(m+1)}-1} x_{t_{(m+1)}-2} \ldots x_{t_m}, \\ y_m &= w_{t_m} w_{t_m+1} \ldots w_{t_{(m+1)}-1}, \end{aligned}$$

for $1 \leq m \leq |D|$. Thus

$$v_{j_{(t_m)}} = y_m v_{j_{(t_{(m+1)})}} z_m,$$

for all $1 \leq m \leq |D|$. By Lemma 4, $y_m, z_m \in G_S$, for all $1 \leq m \leq |D|$. Therefore the elements $v_{j_{(t_m)}}$ induce a chain C of length $|D|$ in $\mathrm{Div}(V)^U$, for $1 \leq m \leq |D|$. The graph D embeds in C, by Lemma 14, and hence in $\mathrm{Div}(T)^U$.

A similar argument shows that if $\mathrm{Div}(V)^S \cong D_\infty$, then $D \subsetneq \mathrm{Div}(T)^U$.

Thus, in all cases U is D-saturated. □

The following theorem describes all monomial matrix epigroups that are D-saturated.

Theorem 79 *Let S be a subepigroup of the monomial matrix semigroup $M_n(G)$, where G is an infinite semigroup and $M_n(G)$ has the finite ideal series*

$$\{0\} = M_0 \subset M_1 \subset \ldots \subset M_n = M_n(G)$$

such that $M_j/M_{j-1} \cong \mathcal{M}(G_j, I_j, \Lambda_j, P_j)$, for all $1 \leq j \leq n$. Let G_S be the set of all elements contained in subgroups of S, and let E_S be the set of all idempotents of S. For each $1 \leq j \leq n$, let

$$\begin{aligned}
G_{I_j} &= \{i \in I_j \mid G^j_{i*} \cap S \neq \emptyset, R^S_{g^j_{i\lambda}} \not\leq R^S_e, \forall g^j_{i\lambda} \in G^j_{i*} \cap S, e \in E_S\}, \\
G_{\Lambda_j} &= \{\lambda \in \Lambda_j \mid G^j_{*\lambda} \cap S \neq \emptyset, L^S_{g^j_{i\lambda}} \not\leq L^S_e, \forall g^j_{i\lambda} \in G^j_{*\lambda} \cap S, e \in E_S\}, \text{ and} \\
T_j &= \{g^j_{i\lambda} \in S \mid i \in G_{I_j}, \lambda \in G_{\Lambda_j}\}.
\end{aligned}$$

Then the following statements are equivalent:

(i) *S is divisibility D-saturated;*

(ii) *for each subgroup $G^j_{i\lambda}$ of M, $G^j_{i\lambda} \cap S$ is empty or is a subgroup of S, S contains a finite number of maximal completely 0-simple semigroups, $|\cup_{j=1}^n T_j|$ is finite, the union of all subgroups of S is infinite and, for each subset $G^j_{i\lambda} \cap S$, not contained in a completely 0-simple subsemigroup, G_S induces a congruence ρ on $\langle G_S \cup (G^j_{i\lambda} \cap S)\rangle$ such that $(G^j_{i\lambda} \cap S)/\rho$ is finite;*

(iii) *$S/\mathcal{J}^S$ is finite.*

Proof. (i)⇒(ii): If $G^j_{i\lambda}$ is a subgroup of M, then $G^j_{i\lambda} \cap S$ is either empty or a subgroup of S, by Lemma 73.

The set $(M_j/M_{j-1}) \cap S$ is a subepigroup of M_j/M_{j-1}. Lemma 72 states that $(M_j/M_{j-1}) \cap S$ contains a finite number of maximal completely 0-simple semigroups, for all $1 \leq j \leq n$. Therefore S contains a finite number of maximal completely-0-simple semigroups. Example 74 shows that $(M_j/M_{j-1}) \cap S$ need not contain completely 0-simple semigroups for all j.

Denote by $C_1^j, C_2^j, \ldots, C_{m_j}^j$ the maximal completely 0-simple semigroups in $(M_j/M_{j-1}) \cap S$. Put

$$\begin{aligned} I_{C_k^j} &= \{i \in I_j \mid \exists g_{i\lambda}^j \in C_k^j\}, \\ \Lambda_{C_k^j} &= \{\lambda \in \Lambda_j \mid \exists g_{i\lambda}^j \in C_k^j\}. \end{aligned}$$

Suppose that $|T_j|$ is infinite, for some $1 < j \leq n$. By Corollary 21, there exists an infinite countable subset $T = t_1, t_2, \ldots$ of T_j such that the subgraph H of $\mathrm{Div}(S)$ induced by the elements of T is null, or isomorphic to A_∞, D_∞ or K_∞.

The first case is impossible, because it contradicts (i).

If $H \cong A_\infty$ or $H \cong K_\infty$, then there exist elements $a_p, b_p \in S^1$ such that $t_p = a_p t_{p+1} b_p$, for all $p \in \mathbb{Z}^+$. Therefore the infinite product $a_1 a_2 \ldots$ is nonzero.

Let $A = S^1$ and define a map $\phi : A^+ \to N$, by $\phi(g_{i\lambda}^j) = (j, i, \lambda)$ and $\phi(1) = 1$. The set N is finite. By Lemma 77, $a_1 a_2 \ldots a_{R(2,R(2,3,|N|)+1,|N|)}$ contains a factor

$$w_1 w_2 \ldots w_{R(2,3,|N|)}$$

such that $\phi(w_p) = \phi(w_q) = \phi(w_p w_q)$, for all $1 \leq p \leq q \leq R(2,3,n)$. Let

$$\begin{aligned} w_p &= a_{j_p} a_{j_p+1} \ldots a_{j_{p+1}-1}, \qquad (5.8) \\ x_p &= b_{j_{p+1}-1} b_{j_{p+1}-2} \ldots b_{j_p}. \qquad (5.9) \end{aligned}$$

If $\phi(w_p) \neq 1$, then w_p is contained in a subgroup of M. This means that w_p belongs to a subgroup of S, by Lemma 73, and so $e_{w_p} \in G_S$. Since $t_{j_p} = w_p t_{j_{p+1}} x_p$, we get $e_{w_p} t_{j_p} = t_{j_p}$. Hence $R_{e_{w_p}}^S \geq R_{t_{j_p}}^S$, a contradiction.

If $\phi(w_p) = 1$, then we know by Lemma 77 that $x_{R(2,3,2)} \ldots x_1$ contains factors z_m and z_{m+1} such that $\phi(z_m) = \phi(z_{m+1}) = \phi(z_{m+1} z_m)$. Further, $\phi(z_m), \phi(z_{m+1}) \neq 1$, since $\phi(w_m) = \phi(w_{m+1}) = 1$. For $k = m, m+1$, let

$$z_k = x_{r_{(k+1)}-1} x_{r_{(k+1)}-2} \ldots x_{r_k}. \qquad (5.10)$$

Then $t_{j_{(rm)}} = t_{j_{(r_{(m+1)})}} z_k$, and so $t_{j_{(rm)}} = t_{j_{(rm)}} e_{z_k}$. The element $e_{z_k} \in G_S$, by Lemma 73, and so $L^S_{e_{z_k}} \geq L^S_{t_{j_{(rm)}}}$. This contradicts the definition of T again. It follows that $H \ncong A_\infty$ and $H \ncong K_\infty$.

Similar arguments show that $H \ncong D_\infty$. Therefore we conclude that $|T_j|$ is finite, and hence that $|\cup_{j=1}^n T_j|$ is finite.

At least one $(M_j/M_{j-1}) \cap S$ is infinite, for $1 \leq j \leq n$, since $M_n(G)$ has a finite ideal series. Consider the infinite subsemigroup $(M_k/M_{k-1}) \cap S$ such that $(M_j/M_{j-1}) \cap S$ is finite, for $k < j \leq n$. Then $M_k \cap S$ is D-saturated, by Lemma 75. The quotient semigroup $M_k/M_{k-1} \cap S$ is D-saturated, by Lemma 56. The union of all subgroups of elements of $M_k/M_{k-1} \cap S$ is infinite, by Lemma 72. Hence the union of all subgroups of S is infinite.

Suppose that $G^j_{i\lambda} \cap S$ is infinite and is not contained in a completely 0-simple subsemigroup of $(M_j/M_{j-1}) \cap S$. The subsemigroup

$$W = \langle G_S \cup (G^j_{i\lambda} \cap S) \rangle$$

is D-saturated, by Lemma 78. Let γ be the equivalence on W with all subgroups of S as equivalence classes, and put $\rho = \gamma^\#$.

If $(G^j_{i\lambda} \cap S)/\rho$ is infinite, then there exists an infinite subset T of $G^j_{i\lambda} \cap S$ such that no two elements belong to the same ρ-class.

Corollary 21 states that $\mathrm{Div}(W)$ contains an infinite subgraph G with vertex set a subset of T such that G is null, or isomorphic to A_∞, D_∞ or K_∞. Again, the first case is impossible by (i).

If $G \cong A_\infty$ or $G \cong K_\infty$, then there exists a countable subset $t_1, t_2, \ldots$ of T such that $t_p = a_p t_{p+1} b_p$, for $a_p, b_p \in W^1$ and $p \geq 1$. Then $a_1 a_2 \ldots \neq 0$. Consider the free semigroup A^+, where $A = W^1$, and define $\phi : A^+ \to N$, as before by $\phi(g^j_{i\lambda}) = (j, i, \lambda)$ and $\phi(1) = 1$. By Lemma 77, the product $a_1 a_2 \ldots a_{R(2,R(2,3,|N|)+1,|N|)}$ contains a factor

$$w_1 w_2 \ldots w_{R(2,3,|N|)}$$

such that $\phi(w_p) = \phi(w_q) = \phi(w_p w_q)$, for all $1 \leq p \leq q \leq R(2,3,n)$. Define w_p, x_p and z_k as in equations (5.8)–(5.10), and y_k by

$$y_k = w_{r_k} w_{r_k+1} \ldots w_{r_{(k+1)}-1}.$$

Then

$$\begin{aligned} t_{j_{(rm)}} &= y_m t_{j_{(r_{(m+1)})}} z_m, & (5.11) \\ t_{j_{(r_{(m+1)})}} &= y_{m+1} t_{j_{(r_{(m+2)})}} z_{m+1}. & (5.12) \end{aligned}$$

If $\phi(z_m) \neq 1$, then z_m and z_{m+1} belong to the same subgroup in S, by Lemma 4. Similarly, either $\phi(y_m) = \phi(y_{m+1}) = 1$, or y_m and y_{m+1} belong to the same subgroup in S.

If $\phi(z_m) \neq 1$ and $\phi(y_m) \neq 1$, then $e_{y_m} t_{j(r_{(m+1)})} e_{z_m} = t_{j(r_{(m+1)})}$, by (5.12). Therefore

$$(y_m t_{j(r_{(m+1)})} z_m, e_{y_m} t_{j(r_{(m+1)})} e_{z_m}) = (t_{j(r_m)}, t_{j(r_{(m+1)})}) \in \rho,$$

a contradiction. If $\phi(y_m) = 1$ and $\phi(z_m) \neq 1$, or if $\phi(z_m) = 1$ and $\phi(y_m) \neq 1$, then a similar argument shows that $(t_{j(r_m)}, t_{j(r_{(m+1)})}) \in \rho$, a contradiction again.

Thus, in all cases a contradiction arises. Therefore $G \not\cong A_\infty$ and $G \not\cong K_\infty$. The dual argument shows that $G \not\cong D_\infty$. Hence we conclude that $(G^j_{i\lambda} \cap S)/\rho$ is finite in W.

(ii)$\Rightarrow$(iii): Evidently, $|(G^j_{i\lambda} \cap S)/\mathcal{J}^S| = 1$, if $G^j_{i\lambda} \cap S$ belongs to a completely 0-simple semigroup C^j_k. Since T_j is finite, $|(G^j_{i\lambda} \cap S)/\mathcal{J}^S|$ is finite, for all $i \in G_{I_j}, \lambda \in G_{\Lambda_j}$ and $1 \leq j \leq n$.

Suppose that $G^j_{i\lambda} \cap S$ is infinite and is not contained in a completely 0-simple semigroup C^j_h of S. Select two elements $a, b \in G^j_{i\lambda} \cap S$ such that $(a, b) \in \rho$. By Lemma 2, there exists a sequence

$$a = z_1 \to z_2 \to \ldots \to z_n = b$$

of elementary γ-transitions connecting a to b. Hence there exist elements $x, y, c, d \in W^1$ such that $z_1 = xcy$ and $z_2 = xdy$. By the definition of γ, the elements c, d belong to some subgroup $G^k_{\ell\mu} \cap S$ of W.

Put $x = x_1 x_2 \ldots x_{n_1}$ and $y = y_1 y_2 \ldots y_{n_2}$, where $x_p, y_q \in G_S \cup (G^j_{i\lambda} \cap S)$, for $1 \leq p \leq n_1, 1 \leq q \leq n_2$. If $x_p, y_q \in G_S$, for all $1 \leq p \leq n_1, 1 \leq q \leq n_2$, then by [21], Corollary 5.1.3, the term $(y_{n_2})^{-1}(y_{n_2-1})^{-1} \ldots y_1^{-1} c^{-1} (x_{n_1})^{-1} \ldots (x_1)^{-1}$ is the inverse of z_1 and is contained in W. This implies that $G^j_{i\lambda} \cap S$ is contained within C^j_h, for some h, a contradiction. Therefore either $x_p \in G^j_{i\lambda} \cap S$ or $y_q \in G^j_{i\lambda} \cap S$, for some $1 \leq p \leq n_1, 1 \leq q \leq n_2$.

If $x_p \in G^j_{i\lambda} \cap S$, then $x_p x_{p+1} \ldots x_n c y$ is also contained in $G^j_{i\lambda} \cap S$. Therefore by [21], Lemma 2.2.4, $w_c = x_{p+1} \ldots x_n c y$ may be associated with a bijection ϕ_{w_c} of $G^j_{i\lambda} \cap S$ onto itself defined by $\phi_{w_c}(g) = g w_c$. Then w_c is not nil, and so there exists an integer $1 \leq r \leq 2^n$ such that w^r_c belongs to a subgroup of S. The map $\phi_{w^r_c} : G^j_{i\lambda} \cap S \to G^j_{i\lambda} \cap S$, defined by $\phi_{w^r_c}(g) = g w^r_c$ is also a bijection.

Therefore $ge_{w_c^r} = g$, for all $g \in G_{i\lambda}^j \cap S$, and in particular, $x_p e_{w_c^r} = x_p$. Hence $z_1 w_c^{r-1} = x_1 \dots x_p w_c^r$, and so $z_1 w_c^{r-1}(w_c^r)^{-1} = x_1 \dots x_p$.

Let $w_d = x_{p+1} \dots x_n dy$. Then the corresponding arguments apply and $z_2 w_d^{r-1}(w_d^r)^{-1} = x_1 \dots x_p$. Thus we see that $z_1 = z_2 w_d^{r-1}(w_d^r)^{-1} w_c$ and that $z_2 = z_1 w_c^{r-1}(w_c^r)^{-1} w_d$. Therefore $z_1 \mathcal{J}^W z_2$.

If $y_q \in G_{i\lambda}^j \cap S$, for some $1 \le q \le n_2$, then the dual argument proves that $z_1 \mathcal{J}^W z_2$.

Easy induction on z_k shows that $a \mathcal{J}^W b$. Hence $\rho \subseteq \mathcal{J}^W$, when restricted to $G_{i\lambda}^j \cap S$.

Clearly, $\mathcal{J}^W \subseteq \mathcal{J}^S$, and so

$$|(G_{i\lambda}^j \cap S)/\mathcal{J}^S| \le |(G_{i\lambda}^j \cap S)/\mathcal{J}^W| \le |(G_{i\lambda}^j \cap S)/\rho| < \infty.$$

Thus we see that $|(G_{i\lambda}^j \cap S)/\mathcal{J}^S|$ is finite, for all $i \in I_j, \lambda \in \Lambda_j$ and $1 \le j \le n$. Since I_j and Λ_j are finite, for all $1 \le j \le n$,

$$|S/\mathcal{J}^S| \le \sum_{j=1}^{n} (\sum_{i \in I_j} (\sum_{\lambda \in \Lambda_j} |(G_{i\lambda}^j \cap S)/\mathcal{J}^S|)) < \infty.$$

The implication (iii)$\Rightarrow$(i) follows from Lemma 58. $\square$

Chapter 6

Annihilator D-saturated semigroups

This chapter is devoted to a combinatorial property defined in terms of annihilator graphs. The *annihilator graph* $\mathrm{Ann}(S)$ of a semigroup $S = S^0$ has vertex set S^0 and edge set $\{(u, v) \in S^0 \times S^0 \mid uv = 0, u \neq v\}$. For a finite graph D, a semigroup S is *annihilator D-saturated* if and only if, for every infinite subset T of S, the annihilator graph $\mathrm{Ann}(S)$ has a subgraph isomorphic to D with all vertices being in T.

We begin by showing that the class of all annihilator D-saturated semigroups is closed under subsemigroups and homomorphic images, but not direct products. The first main theorem describes all commutative semigroups S and finite graphs D such that S is D-saturated. Necessary and sufficient conditions are then given for all completely 0-simple semigroups which possess the same combinatorial property. This result is used to characterise all linear D-saturated semigroups.

6.1 Properties of annihilator D-saturated semigroups

Lemma 80 *If S is annihilator D-saturated, then all subsemigroups H of S are annihilator D-saturated, too.*

Proof. Take any infinite subset T of H. Since S is annihilator D-saturated, D embeds in the subgraph G of $\mathrm{Ann}(S)$ with all vertices being in T. Then H is annihilator D-saturated, since $G \subseteq \mathrm{Ann}(H)$. □

Lemma 81 *Let S be an infinite semigroup that is annihilator D-saturated, and let ρ be a congruence on S. Then S/ρ is annihilator D-saturated.*

Proof. If S/ρ is finite, then the result is vacuously true. Therefore we may assume that S/ρ is infinite.

Denote by 0 and θ the zeros of S and S/ρ, respectively. Obviously, $\rho 0 = \theta$.

Take any infinite subset T of S/ρ. For each $t_i \in T$, choose a representative $s_i \in S$ such that $\rho s_i = t_i$, and let U be the set of these representatives. Then $\rho^{\natural} : U \to T$ is a bijection. Hence U is infinite, and so D embeds in the subgraph G induced by the elements of U.

Suppose that $(s_i, s_j) \in E(G)$. Then $s_i s_j = 0$. Therefore

$$t_i t_j = \rho s_i \rho s_j = \rho(s_i s_j) = \rho 0 = \theta,$$

and so $(t_i, t_j) \in E(S/\rho)$. Therefore G embeds in $\mathrm{Ann}(S/\rho)$. Hence D embeds in $\mathrm{Ann}(S/\rho)$. Thus S/ρ is D-saturated. □

Example 82 shows that the family of annihilator D-saturated semigroups is not closed under direct products.

Example 82 Let $S = \{s_i \mid i \in \mathbb{Z}^+\}$ be the commutative semigroup with zero 0, where multiplication is defined by the rules

$$s_i s_j = \begin{cases} s_j & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Clearly, S is annihilator D-saturated, but the direct product $S \times S$ is not, since the elements $(s_1, s_1), (s_1, s_2), (s_1, s_3), \ldots$ induce a null subgraph in $\mathrm{Ann}(S \times S)$.

Thus the class of annihilator D-saturated semigroups is not a variety, and resembles that of power D-saturated semigroups as far as this property is concerned.

On the other hand, there is an essential difference between annihilator D-saturated semigroups and semigroups possessing other combinatorial properties considered in this thesis. For every semigroup S, the power graph $\mathrm{Pow}(S)$, the divisibility graph $\mathrm{Div}(S)$ and the Cayley graph $\mathrm{Cay}(S,T)$ are transitive. Example 83 shows that $\mathrm{Ann}(S)$ does not always have to be transitive.

Example 83 Let S be the commutative semigroup with zero 0, generators a, b, c, and relations $a = a^2, b = b^2, c = c^2, ab = bc = 0, (ac)^2 = ac$. Then $(a, b), (b, c) \in E(\mathrm{Ann}(S))$, but $(a, c) \notin E(\mathrm{Ann}(S))$. Thus the binary relation of the annihilator graph of S is not transitive.

6.2 Commutative semigroups

If T is a subset of a semigroup S and $0 \notin T^2$, then the elements of T induce a null subgraph in $\mathrm{Ann}(S)$. The following example demonstrates that the converse is not true.

Example 84 Let $S = \{0, t, s_1, s_2, \ldots\}$ be the commutative semigroup with multiplication defined by

$$\begin{array}{ll} 0s = s0 = 0, & \text{for all } s \in S, \\ s^2 = 0, & \text{for all } s \in S, \\ s_i t = t s_i = 0, & \text{for all } i \in \mathbb{Z}^+, \\ s_i s_j = t, & \text{for all } i, j \in \mathbb{Z}^+ \text{ and } i \neq j. \end{array}$$

Every subset T of S induces a null subgraph in $\mathrm{Ann}(S)$, but $0 \in T^2$.

The leads to the following definition.

Definition 85 For a subset T of a semigroup S,

$$T^\star = \{uv \in S \mid u, v \in T, u \neq v\}.$$

Theorem 86 *Let $D = (V, E)$ be a finite graph with $E \neq \emptyset$, and let S be an infinite commutative semigroup that has a zero and is a semilattice Y of Archimedean semigroups S_y. Then S is annihilator D-saturated if and only if the following conditions hold:*

(i) *all Archimedean components of S are finite;*

(ii) *all chains in Y are finite;*

(iii) *for every infinite subset T of Y, the set $T^\star$ contains the zero of Y.*

Proof. For any $s \in S$, denote by $\phi(s)$ the element of Y such that $s \in S_{\phi(s)}$. Let 0 stand for the zero of S. Clearly, Y also has a zero θ, and $S_\theta = \{0\}$.

The 'only if' part. Suppose that S is annihilator D-saturated. Since the elements of S_y are not adjacent in $\mathrm{Ann}(S)$, it follows from annihilator D-saturation that S_y is finite, i.e., (i) holds.

Hence each Archimedean component S_y has an idempotent e_y. If Y contains an infinite chain C, then we may assume that the zero of Y is not in C, and therefore all vertices e_y, for $y \in C$, are not adjacent in $\mathrm{Ann}(S)$. This contradiction shows that (ii) holds.

Let T be a subset of Y such that $\theta \notin T^\star$. Then $U = \{\phi^{-1}(t) \mid t \in T\}$ does not contain the zero of S, and $0 \notin U^\star$. Therefore all elements of U are not adjacent in $\mathrm{Ann}(S)$. It follows that T is finite, and so (iii) is satisfied.

The 'if' part. Suppose that all Archimedean components of S are finite, all chains of Y are finite, and Y does not contain infinite subsets T such that $\theta \notin T^\star$. Let U be an infinite subset of S. By (i), we may assume that $0 \notin U$, and different elements of U belong to different Archimedean components of S. For every edge $(s,t) \in E(\mathrm{Ann}(S))$, the reversed edge (t,s) also belongs to $E(\mathrm{Ann}(S))$, since S is commutative

Let Z be the subset $\{\phi(u) \mid u \in U\}$ of Y. We colour a two-element subset $\{i,j\}$ of Z in green, if $\phi^{-1}(i)\phi^{-1}(j) = 0$, and we colour it in red otherwise. By Lemma 20, there exists an infinite subset T of Z such that all two-element subsets of T have the same colour.

If they are all red, then the elements $\phi^{-1}(t), t \in T$, induce a null subgraph in $\mathrm{Ann}(S)$. Therefore $0 \notin (\phi^{-1}(T))^\star$, and so $\theta \notin T^\star$, a contradiction to (iii). On the other hand, if they are all green, then $xy = yx = 0$ for all $x, y \in \phi^{-1}(T)$, $x \neq y$. Therefore the set $\{\phi^{-1}(t) \mid t \in T\} \subseteq U$ induces an infinite complete symmetric subgraph in $\mathrm{Ann}(S)$. Clearly, D embeds in U, and so S is annihilator D-saturated. $\square$

6.3 Semigroups with a finite ideal series

Completely 0-simple semigroups

The next lemma describes all annihilator D-saturated completely 0-simple semigroups.

Lemma 87 *Let D be a finite graph with at least one edge, and let M be an infinite completely 0-simple semigroup such that $M = \mathcal{M}^0(G; I, \Lambda; P)$. Then the following are equivalent:*

(i) *M is D-saturated;*

(ii) *G is finite, I and Λ are infinite, and the union of all subgroups of M in every $\mathcal{L}$-class and every $\mathcal{R}$-class of M is finite;*

(iii) *for every infinite subset T of M, $0 \in T^\star$.*

Proof. (i) $\Rightarrow$ (ii): Consider any maximal subgroup $G_{i\lambda}$ of M. Evidently, the elements of $G_{i\lambda}$ induce a null subgraph in $\mathrm{Ann}(M)$. If $G_{i\lambda}$ is infinite, then $\mathrm{Ann}(M)$ contains an infinite null subgraph. Since D has edges, it does not embed in this subgraph, and M is not D-saturated. This contradiction shows that every maximal subgroup of M is finite.

If Λ and I are both finite, then M contains an infinite maximal subgroup, a contradiction as seen in the preceding paragraph.

If Λ is finite and I is infinite, then by [21], Theorem 3.2.3, the set $L = \{i \in I \mid G_{i\lambda}$ is a subgroup of $M\}$ is infinite, for some fixed $\lambda \in \Lambda$. Then the set $\{g_{i\lambda} \in G_{*\lambda} \mid i \in L\}$ is infinite, and induces a null subgraph in $\mathrm{Ann}(M)$, by Lemma 4. This contradicts (i) again. The dual argument leads to a contradiction if I is finite and Λ is infinite. Thus we see that I and Λ are both infinite.

If the union of all subgroups is infinite for some fixed $\mathcal{L}$-class $G_{*\lambda}$, then we see that the set $\{g_{i\lambda} \in G_{*\lambda} \mid (g_{i\lambda})^2 \neq 0\}$ is infinite and induces a null subgraph in $\mathrm{Ann}(M)$, by Lemma 4. This contradicts (i) again, and shows that the union of all subgroups of M in every $\mathcal{L}$-class of M is finite.

The dual argument shows that union of all subgroups of M in every $\mathcal{R}$-class G_{i*} of M is finite.

We know from Lemma 5(iv) that all $\mathcal{H}$-classes of M have the same cardinality. Thus M contains infinitely many $\mathcal{H}$-classes, and so at least one of I or Λ is infinite. If Λ is finite, then by [21], Theorem 3.2.3, $G_{*\lambda}$ contains infinitely many $\mathcal{H}$-classes which are subgroups, for some $\lambda \in \Lambda$. This contradicts the preceding paragraphs. Therefore Λ is infinite. The dual argument shows that I is also infinite.

Every $\mathcal{L}$-class contains a subgroup isomorphic to G, by Lemma 5(viii). Since Λ is infinite, the union of all subgroups of M is infinite.

(ii) $\Rightarrow$ (iii): Take any infinite subset T of M. Clearly, T contains either an infinite subset U whose elements are all pairwise $\mathcal{L}$-equivalent, or an infinite subset U whose elements are all pairwise $\mathcal{R}$-equivalent, or an infinite subset U where no two elements are $\mathcal{R}$-equivalent or $\mathcal{L}$-equivalent.

If the elements of U are pairwise $\mathcal{L}$-equivalent, then there exist $g_{i\lambda}, g_{j\lambda}$ in U such that $g_{i\lambda}^2 = g_{j\lambda}^2 = 0$. By [21], Lemma 3.2.7, $g_{i\lambda}g_{j\lambda} = 0$, and so $0 \in T^\star$. The same conclusion follows in the case where all elements of U are pairwise $\mathcal{R}$-equivalent.

If all elements of U are contained in distinct $\mathcal{L}$-classes and $\mathcal{R}$-classes, then put $U = g_{i_1\lambda_1}, g_{i_2\lambda_2}, \ldots$, where $i_k \in I, \lambda_k \in \Lambda$ and $i_k \neq i_j, \lambda_k \neq \lambda_j$, for $k \neq j$. Since I and Λ are infinite, there exists an element $g_{i_1\lambda_k}$ such that $g_{i_1\lambda_k}^2 = 0$. Then $g_{i_1\lambda_1}g_{i_k\lambda_k} = 0$, by [21], Lemma 3.2.7. Thus $0 \in T^\star$ in this case, too.

(iii) $\Rightarrow$ (i): Take any infinite subset T of M.

First, suppose that T contains a countably infinite subset U of pairwise $\mathcal{L}$-equivalent elements. By Corollary 21, there exists an infinite subset W of U such that the subgraph H of $\mathrm{Ann}(M)$ induced by the elements of W is null, or isomorphic to A_∞, D_∞ or K_∞.

If H is null, then $0 \notin W^\star$, a contradiction.

If $H \cong A_\infty$, then index the elements of W such that $w_i w_j = 0$, for all $w_i, w_j \in W$ and $1 \leq i \leq j$. Pick any three elements w_i, w_j and w_k, where $i < j < k$. Since $w_i \mathcal{L} w_k$, there exists an element $m \in M$ such that $w_k = m w_i$. Then $w_k w_j = m w_i w_j = 0$. That is, the edge $(w_k, w_j) \in E(\mathrm{Ann}(M))$. This contradiction shows that $H \ncong A_\infty$.

A similar argument implies that $H \ncong D_\infty$.

Therefore W contains a subset of elements inducing an infinite complete symmetric graph K_∞ in $\mathrm{Ann}(M)$. Then D embeds in this subgraph, and hence in $\mathrm{Ann}(M)$. Thus M is D-saturated in this case.

The dual proof leads to the same conclusion, if T contains a countably infinite subset of pairwise $\mathcal{R}$-equivalent elements.

Next, suppose that T contains a countably infinite subset U such that no two elements are $\mathcal{L}$-equivalent or $\mathcal{R}$-equivalent. Applying Corollary 21 again, we see that U contains an infinite subset $W = \{g_{i_k\lambda_k} \mid k \in \mathbb{Z}^+\}$ such that the subgraph H of $\mathrm{Ann}(M)$ with all vertices being in W is null, or isomorphic to A_∞, D_∞ or K_∞.

If H is null, then $0 \notin W^\star$, contradicting (iii).

If $H \cong A_\infty$, then we may assume that $g_{i_j\lambda_j} g_{i_k\lambda_k} \neq 0$, for $1 \le k < j$. By [21], Lemma 3.2.7, $G_{i_1\lambda_k}$ is a subgroup of M, for all $k \in \mathbb{Z}^+$. Then the set $X = \{g_{i_1\lambda_k} \mid k \in \mathbb{Z}^+\}$ is infinite and $0 \notin X^\star$, which contradicts (iii). Thus $H \ncong A_\infty$.

A similar argument shows that $H \ncong D_\infty$.

Therefore we deduce that $H \cong K_\infty$. The graph D embeds in this subgraph, and hence in $\mathrm{Ann}(M)$. Thus M is D-saturated in this case, too. □

Semisimple semigroups

Consider a semisimple semigroup $M = M^0$ with principal series

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M^0.$$

If M is D-saturated, then clearly M_j/M_{j-1} is D-saturated, for all $1 \le j \le n$. Example 88 shows that the converse is not true.

Example 88 Let G be an infinite group, $I = \{0, 1\}$, D a finite graph with at least one edge, and let S be the semigroup with zero θ and all elements of

$G \times I$, where multiplication is defined by

$$(g,i)(h,j) = \begin{cases} (gh, 0) & \text{if } i = 1 \text{ and } j = 1, \\ \theta & \text{otherwise.} \end{cases}$$

The semigroup S has a ideal series, namely

$$\theta \subset G \times \{0\} \cup \theta \subset S.$$

The Rees quotient semigroups $(G \times \{0\} \cup \theta)/\theta$ and $S/(G \times \{0\} \cup \theta)$ are both D-saturated, but S is not, since the elements in $G \times \{1\}$ induce a null subgraph in $\mathrm{Ann}(S)$.

Theorem 89 *Let $M = M^0$ be an infinite semigroup admitting a principal series*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M^0,$$

where each factor M_j/M_{j-1} is isomorphic to the completely 0-simple semigroup $\mathcal{M}^0(G_j; I_j, \Lambda_j; P_j)$, for all $1 \leq j \leq n$. Let D be a finite graph with at least one edge. Then the following are equivalent:

(i) *M is annihilator D-saturated;*

(ii) *the union of all subgroups of M is infinite, the union of all subgroups in every $\mathcal{L}$-class and every $\mathcal{R}$-class of M_j/M_{j-1} is finite, and $m^2 = 0$ for all but a finite number of elements m in every $\mathcal{L}$-class and every $\mathcal{R}$-class of M_j/M_{j-1};*

(iii) *for every infinite subset T of M, $0 \in T^{\star}$.*

Proof. (i)$\Rightarrow$(ii): There exists at least one j such that M_j is infinite, since M is infinite. Then M_j is D-saturated, by Lemma 80. Therefore M_j/M_{j-1} is D-saturated, by Lemma 81. By Lemma 87, the union of all subgroups in M_j/M_{j-1} is infinite. Hence the union of all subgroups of M is infinite.

Lemma 87 also tells us the that the union of all subgroups of M_j/M_{j-1} in every $\mathcal{L}$-class and in every $\mathcal{R}$-class of M_j/M_{j-1} is finite.

Suppose to the contrary that the set

$$L = \{m \in G^j_{*\lambda} \mid m^2 \neq 0\},$$

is infinite, for some $\mathcal{L}$-class $G^j_{*\lambda}$ of M_j/M_{j-1}. For $m_k, m_\ell \in L$, we know that $m_k = tm_\ell$, for some $t \in M$. Therefore the element $tm_\ell m_k = m_k^2 \neq 0$, and so $m_\ell m_k \neq 0$. Similarly, $m_k m_\ell$ is nonzero. Hence the elements of L induce an infinite null subgraph in $\mathrm{Ann}(M)$. The graph D does not embed in this subgraph, contradicting (i). Thus L is finite, and so $m^2 = 0$, for all but a finite number of elements m in each $\mathcal{L}$-class of M_j/M_{j-1}.

The dual argument shows that $m^2 = 0$, for all but a finite number of elements m in each $\mathcal{R}$-class of M_j/M_{j-1}.

(ii) $\Rightarrow$ (iii): Take any infinite subset T of M. We know that T contains either an infinite subset U whose elements are all pairwise $\mathcal{L}$-equivalent, or an infinite subset U whose elements are all pairwise $\mathcal{R}$-equivalent, or an infinite subset U, where no two elements of U are $\mathcal{R}$-equivalent or $\mathcal{L}$-equivalent.

If T contains an infinite subset U of pairwise $\mathcal{L}$-equivalent elements, then we can find two elements $m_k, m_\ell \in U$ such that $m_k^2 = m_\ell^2 = 0$. By the definition of U, there exists $a \in M$ such that $m_k = am_\ell$. Then $m_k m_\ell = am_\ell^2 = 0$, and so $0 \in T^\star$.

The dual argument applies if T contains an infinite subset of pairwise $\mathcal{R}$-equivalent elements.

If all elements of U are contained in distinct $\mathcal{L}$-classes and $\mathcal{R}$-classes, then for some $1 \leq j \leq n$, there exists an infinite countable subset $V = g^j_{i_1\lambda_1}, g^j_{i_2\lambda_2}, \ldots$ of $U \cap (M_j \setminus M_{j-1})$, where $i_k \in I_j, \lambda_k \in \Lambda_j$ and $i_p \neq i_q, \lambda_p \neq \lambda_q$, for $p \neq q$. Pick an element $g^j_{i_k\lambda_k} \in V$. Since the set $\{m \in G^j_{i_k*} \mid m^2 \neq 0\}$ is finite, there exists $g^j_{i_\ell\lambda_\ell} \in V$ such that $(g^j_{i_k\lambda_\ell})^2 = 0$. Then $g^j_{i_k\lambda_k} = g^j_{i_k\lambda_\ell}a$ and $g^j_{i_\ell\lambda_\ell} = bg^j_{i_k\lambda_\ell}$, for some $a, b \in M$. Thus $g^j_{i_\ell\lambda_\ell}g^j_{i_k\lambda_k} = b(g^j_{i_k\lambda_\ell})^2a = 0$, and so $0 \in T^\star$ in this case, too.

(iii) $\Rightarrow$ (i):Take any infinite subset T of M.

If T contains an infinite subset U of pairwise $\mathcal{L}$-equivalent elements, then the argument used in the proof of Lemma 87 (iii) shows that $K_\infty \subsetneq \mathrm{Ann}(M)$. The graph D embeds K_∞, and hence in $\mathrm{Ann}(M)$. Therefore M is D-saturated in this case.

The same conclusion is drawn if T contains an infinite subset U of pairwise $\mathcal{R}$-equivalent elements.

If all elements of U are contained in distinct $\mathcal{L}$-classes and $\mathcal{R}$-classes, then there exists a subset $V = g^j_{i_1\lambda_1}, g^j_{i_2\lambda_2}, \ldots$ of U, as defined in (ii).

We have just shown that the sets $R_1 = \{g \in G^j_{i_1*} \cap S \mid g^2 \neq 0\}$ and $L_1 = \{g \in G^j_{*\lambda_1} \cap S \mid g^2 \neq 0\}$ are finite. It follows that

$$V_1 = \{g^j_{i_k\lambda_k} \in V \setminus g^j_{i_1\lambda_1} \mid (G^j_{i_1\lambda_k})^2 = (G^j_{i_k\lambda_1})^2 = 0\}$$

is infinite. For each element $g^j_{i_k\lambda_k} \in V_1$, there exist $c_k, d_k, g^j_{i_k\lambda_1} \in M$ such that $g^j_{i_1\lambda_1} = c_k g^j_{i_k\lambda_1}$ and $g^j_{i_k\lambda_k} = g^j_{i_k\lambda_1} d_k$. Therefore $g^j_{i_1\lambda_1} g^j_{i_k\lambda_k} = c_k(g^j_{i_k\lambda_1})^2 d_k = 0$. Similarly, $g^j_{i_1\lambda_1} g^j_{i_k\lambda_k} = 0$, and so $g^j_{i_1\lambda_1} V_1 = V_1 g^j_{i_1\lambda_1} = 0$.

The same reasoning in the preceding paragraphs can now be applied to V_1. That is, for any element $g^j_{i_p\lambda_p} \in V_1$, there exists an infinite subset V_2 of $V_1 \setminus g^j_{i_p\lambda_p}$ such that $g^j_{i_p\lambda_p} V_2 = V_2 g^j_{i_p\lambda_p} = 0$. The process proceeding in the same way is obviously infinite, and so there exists an infinite sequence $g^j_{i_1\lambda_1}, g^j_{i_p\lambda_p}, \ldots$ that induces a complete symmetric subgraph in $\mathrm{Ann}(M)$. The graph D embeds in this subgraph, and hence in $\mathrm{Ann}(M)$. Thus M is D-saturated in this case, too. □

6.4 Matrix semigroups

In the next section necessary and sufficient conditions are given for when a monomial matrix semigroup $M_n(G)$ is annihilator D-saturated. The proof relies on the structure of $M_n(G)$, described in Lemma 24.

Monomial matrix semigroups

Theorem 90 *Let S be an infinite matrix semigroup of $M_n(G)$, where $M_n(G)$ has ideal series*

$$\{0\} = M_0 \subset M_1 \subset \ldots \subset M_n = M_n(G),$$

and each factor M_j/M_{j-1} is isomorphic to the completely 0-simple semigroup $\mathcal{M}^0(G_j; I_j, \Lambda_j; P_j)$, for all $1 \leq j \leq n$. Let G_S be the set of all elements contained in subgroups of S, and let D be a finite graph with at least one edge. Then the following statements are equivalent:

(i) *S is annihilator D-saturated;*

(ii) *S is periodic, G_S is finite, and $s^2 = 0$, for all but a finite number of elements of S;*

(iii) *for every infinite subset T of S, $0 \in T^\star$.*

Proof. (i) $\Rightarrow$ (ii): If S contains an element s of infinite order, then the vertices $s, s^2, s^3, \ldots$ are not adjacent in $\mathrm{Ann}(S)$. Since D has edges, we see that the subgraph induced by the vertices $s, s^2, s^3, \ldots$ does not contain a subgraph isomorphic to D, a contradiction. Thus S is periodic.

Suppose to the contrary that G_S is infinite. Each Rees factor M_j/M_{j-1} contains a finite number of $\mathcal{H}$-classes, and hence a finite number of maximal subgroups. Therefore G_S contains an infinite maximal subgroup H. The subgraph induced by this subgroup is null, and so D does not embed in this subgraph. Therefore S is not D-saturated, contradicting (i). Thus G_S is finite.

Suppose that the set $T = \{s \in S \mid s^2 \neq 0\}$ is infinite. Then $T \cap G^j_{i\lambda}$ is infinite, for some $i \in I_j, \lambda \in \Lambda_j$ and $1 \leq j \leq n$, since I_j, Λ_j are finite, for all $1 \leq j \leq n$. Pick two elements $s_k, s_\ell \in T \cap G^j_{i\lambda}$. There exists $t \in M$ such that $s_k = s_\ell t$. Therefore the element $s_k s_\ell t = s_k^2 \neq 0$, and so $s_k s_\ell \neq 0$. Similarly, $s_\ell s_k$ is nonzero, and so the elements of $T \cap G^j_{i\lambda}$ induce an infinite null subgraph in $\mathrm{Ann}(S)$. This contradicts (i) again, and so we conclude that T is finite.

(ii) $\Rightarrow$ (iii): Take any infinite subset T of S. Then $T \cap G^j_{i\lambda}$ is infinite, for some $i \in I_j, \lambda \in \Lambda_j$ and $1 \leq j \leq n$. Since $T = \{s \in S \mid s^2 \neq 0\}$ is finite, we can find two elements $s_k, s_\ell \in T \cap G^j_{i\lambda}$ such that $s_k^2 = s_\ell^2 = 0$. There exists $a \in M$ such that $s_k = s_\ell a$. Thus $s_k s_\ell = s_\ell^2 a = 0$, and so $0 \in T^\star$.

(iii) $\Rightarrow$ (i): Take any infinite subset T of S. Then $T \cap G^j_{i\lambda}$ is infinite, for some $\mathcal{H}$-class $G^j_{i\lambda}$ of $M_n(G)$. Since $0 \in (T \cap G^j_{i\lambda})^\star$, there exist two distinct elements $a, b \in T \cap G^j_{i\lambda}$ such that $ab = 0$. For any other pair of elements $c, d \in T \cap G^j_{i\lambda}$, there exist $e, f \in M_n(G)$ such that $c = ea$ and $d = bf$. Therefore $cd = eabf = 0$. Similarly, $dc = 0$, and so $\mathrm{Ann}(S)$ contains an infinite complete symmetric subgraph. The graph D embeds in this subgraph, and hence S is D-saturated. $\square$

Annihilator D-saturated linear semigroups are more difficult to describe, since each Rees quotient semigroup can have an infinite sandwich matrix.

Theorem 91 *Let S be an infinite matrix semigroup of $M_n(K)$, where $M_n(K)$ has ideal series*

$$\{0\} = M_0 \subset M_1 \subset \ldots \subset M_n = M_n(K).$$

Let G_S be the set of all elements contained in subgroups of S, and let D be a finite graph with at least one edge. Then S is annihilator D-saturated if and only if the following conditions hold:

(i) *S is periodic;*

(ii) *$s^2 = 0$, for all but a finite number of elements of S;*

(iii) *G_S is finite;*

(iv) *for every infinite subset T of S, $0 \in T^{\star}$.*

Proof. The 'only if' part. Suppose that S is annihilator D-saturated.

The argument used in the proof of Theorem 90(i) demonstrates that S is periodic. Thus (i) holds.

Suppose to the contrary that there exists an infinite subset T of S such that $t^2 \neq 0$, for all $t \in T$. By Corollary 21, there exists a countably infinite subset $U = u_1, u_2, \ldots$ of T such that the subgraph H of $\mathrm{Ann}(S)$ with all vertices of U is null, or isomorphic to either A_∞, D_∞ or K_∞.

If H is null, then S is not D-saturated, a contradiction.

If $H \cong A_\infty$ or $H \cong K_\infty$, then we may assume that

$$u_i u_j = 0, \tag{6.1}$$

for all $u_i, u_j \in U$ and $1 \leq i < j$. The elements of U are contained in a vectorspace (see [55], §4.1) of dimension n^2. Therefore there exists a set $u_{s_1}, u_{s_2}, \ldots, u_{s_{n^2}}$, which spans U. Then

$$u_{s_{n^2+1}} = a_1 u_{s_1} + \ldots + a_{n^2} u_{s_{n^2}},$$

where $a_i \in K$. Hence

$$u_{s_{n^2+1}}^2 = (a_1 u_{s_1} + \ldots + a_{n^2} u_{s_{n^2}}) u_{s_{n^2+1}}. \tag{6.2}$$

The left-hand side of (6.2) is nonzero, since $u_{s_{n^2+1}} \in T$. The right-hand side of (6.2) is zero, by (6.1). This contradiction shows that $H \ncong A_\infty$ and $H \ncong K_\infty$.

If $H \cong D_\infty$, we may assume that $u_i u_j = 0$, for all $1 \leq j < i$. Then

$$u_{s_{n^2+1}} = a_1 u_{s_1} + \ldots + a_{n^2} u_{s_{n^2}}.$$

Hence

$$u_{s_{n^2+1}}^2 = u_{s_{n^2+1}} (a_1 u_{s_1} + \ldots + a_{n^2} u_{s_{n^2}}),$$

which yields the same contradiction.

This contradicts Corollary 21, and so we conclude that T is finite. Therefore (ii) holds.

It follows immediately that G_S is finite, since $s^2 \neq 0$, for all $s \in G_S$. Thus (iii) holds.

Take any infinite subset T of S. Again, Corollary 21 implies that T contains an infinite subset W such that the subgraph H of $\mathrm{Ann}(S)$ induced by the vertices of W is null, or isomorphic to A_∞, D_∞ or K_∞. The first possibility is impossible by our hypothesis. In the remainining cases H contains edges. If $(a, b) \in E(H)$, then $ab = 0$, and so $0 \in T^\star$. Hence (iv) holds.

The 'if' part. Take any infinite subset T of S. Applying Corollary 21, T contains a countably infinite subset $U = u_1, u_2, \ldots$ such that the subgraph H of $\mathrm{Ann}(S)$ induced by the elements of U is null, or isomorphic to A_∞, D_∞ or K_∞.

The first case implies that S is not D-saturated, and so is impossible.

If $H \cong A_\infty$, then we may assume that the elements of U have been indexed such that $u_i u_j = 0$ (and $u_j u_i \neq 0$), for all $1 \leq i < j$. Choose a spanning subset $u_{s_1}, u_{s_2}, \ldots, u_{s_{n^2}}$ of U. We get

$$u_{s_{n^2+2}} = a_1 u_{s_1} + \ldots + a_{n^2} u_{s_{n^2}}, \tag{6.3}$$

for some $a_i \in K$. Hence

$$u_{s_{n^2+2}} u_{s_{n^2+1}} = (a_1 u_{s_1} + \ldots + a_{n^2} u_{s_{n^2}}) u_{s_{n^2+1}}. \tag{6.4}$$

The left-hand side of (6.4) is nonzero. The right-hand side of (6.4) is zero. Therefore $H \not\cong A_\infty$.

If $H \cong D_\infty$, then a similar argument yields the same contradiction. Therefore $H \not\cong D_\infty$.

We deduce from Corollary 21 that $H \cong K_\infty$. The graph D embeds in this subgraph. Hence D embeds in $\mathrm{Ann}(S)$, and S is D-saturated in this case, too. $\square$

6.5 Nilpotent semigroups

In contrast with the cases of commutative, semisimple and matrix semigroups, there exist nilpotent semigroups that are D-saturated, provided D is acyclic, as seen in Example 92.

Example 92 Let $S = \{0, \varepsilon, x_1, x_2, \ldots\}$, where multiplication is defined by the rule:

$$\begin{array}{ll} x_i x_j = \varepsilon, & \text{if } i \leq j, \\ x_i x_j = 0, & \text{if } i > j, \\ s\varepsilon = \varepsilon s = 0, & \text{for all } s \in S, \\ s0 = 0s = 0, & \text{for all } s \in S. \end{array}$$

Then $\mathrm{Ann}(S)$ is isomorphic to D_∞. Hence S is D-saturated if and only if D is acyclic.

Not all nilpotent semigroups are D-saturated, of course, as evidenced in Example 84.

It is difficult to describe annihilator D-saturated semigroups in full generality. However, the next result shows that it suffices to consider only the elements of $S \setminus S^2$ when investigating whether a nilpotent semigroup is annihilator D-saturated or not.

Lemma 93 *An infinite nilpotent semigroup S is annihilator D-saturated if and only D embeds in the subgraph of $Ann(S)$ with all vertices being in $S \setminus S^2 \cup \{0\}$.*

Proof. Denote by n the minimum positive integer such that $S^n = 0$.

The 'only if' part is obvious.

The 'if' part. The proof uses induction, and the base step is the hypothesis, namely, that D embeds in the subgraph of $\mathrm{Ann}(S)$ with all vertices being in $S \setminus S^2 \cup \{0\}$. Assume that D embeds in the subgraphs H_k of $\mathrm{Ann}(S)$ with all vertices being in $S \setminus S^k \cup \{0\}$, for $k = 3, 4 \ldots n-1$.

Now $S \setminus S^n \cup \{0\} = S^{n-1} \cup S \setminus S^{n-1} \cup \{0\}$. Every infinite subset T of $S \setminus S^n \cup \{0\}$ contains an infinite subset U of T such that either $U \subseteq S^{n-1} \cup \{0\}$ or $U \subseteq S \setminus S^{n-1} \cup \{0\}$. The product of every pair of elements in $S^{n-1} \cup \{0\}$ is zero. Therefore D embeds in the subgraph G of $\mathrm{Ann}(S)$ induced by the elements of $S^{n-1} \cup \{0\}$, if $U \subseteq S^{n-1} \cup \{0\}$. When $U \subseteq S \setminus S^{n-1} \cup \{0\}$, the graph D embeds in the subgraph H of $\mathrm{Ann}(S)$ induced by the elements of $S \setminus S^{n-1} \cup \{0\}$, by our induction assumption. Therefore D embeds in the subgraph H_n of $\mathrm{Ann}(S)$ with all vertices being in $S \setminus S^{n-1} \cup \{0\}$. Since $S = S \setminus S^n \cup \{0\}$, we see that S is D-saturated. $\square$

Appendix A

Automata recognised by graph algebras

A.1 Introduction

This appendix is devoted to a connection between automata and a particular class of algebras, called "graph algebras". First, we recall a few definitions of automata theory.

Definition 94 ([22], §2.2.9) A *deterministic finite state automaton* is a quintuple $\mathbf{A} = \mathcal{A}(S, X, \phi, i, T)$, where

S is a finite set of states;
X is a finite set of symbols;
$\phi : S \times X \to S$ is a state transition function;
$i \subseteq S$ is an initial set of states;
$T \subseteq S$ is a set of final states.

A word $x \in X^*$ is said to be *recognised* by an automaton $\mathbf{A}$ if $ix \in T$. The *language* recognised by an automaton is the set of words in X^* which are recognised by $\mathbf{A}$. It is denoted by $L(\mathbf{A})$.

An automaton can be represented diagrammatically, via a labelled directed graph with vertex set S and edge set (s, t), for all $s, t \in S$, where

$sx = t$, for some $x \in X$. The edge (s, t) is labelled by x. This graph is referred to as the *state diagram* of the automaton, and may contain loops and multiple edges. The initial (resp. final) states are represented by incoming (resp., outgoing) arrows. Figure A.1 illustrates the state diagram of an automaton $\mathbf{A} = \mathcal{A}(S, X, \phi, i, T)$, where $S = \{s_1, s_2\}, X = \{a, b\}, i = T = s_1$, and $\phi(s_1, a) = s_2, \phi(s_2, b) = s_1$. The automaton $\mathbf{A}$ recognises the following words: $(ab)^*$.

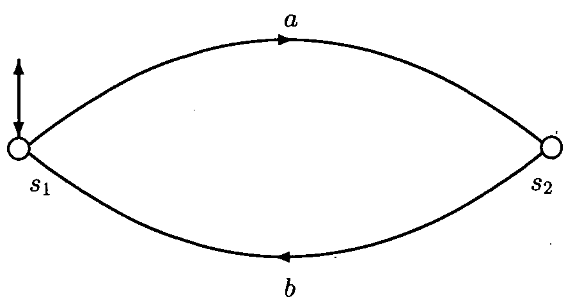


Figure A.1: The state diagram of an automaton.

Schützenberger [51] established the connection between finite automata and finite semigroups, and showed that a finite monoid can be associated with each recognisable language. We need the following definition in order to state this.

Definition 95 ([22], §3.1) A language $L \subseteq X^*$ is *recognised* by a monoid M, if there exists a morphism $\phi : X^* \to M$ and a subset T of M such that $L = \phi^{-1}(T)$.

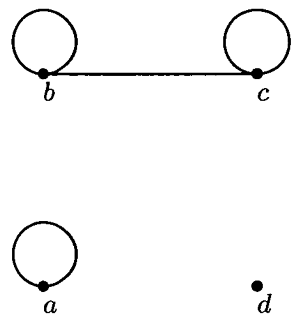
Lemma 96 ([15], Theorem 1.4.2) *Let A be a finite alphabet, and let $L \subseteq A^*$. The following statements are equivalent:*

(i) *L is recognised by a finite automaton;*

(ii) *L is rececognised by a finite monoid.*

Throughout this appendix, a *graph* is finite and undirected without multiple edges, but possibly with loops.

Let $D = (V, E)$ be a graph. The *graph algebra* $\mathrm{Alg}^1(D)$ associated with D is the set $V \cup \{\infty\}$ equipped with multiplication defined by $xy = x$ if $(x, y) \in E$ and $xy = \infty$ otherwise, and $x\infty = \infty x = \infty\infty = \infty$, for all $x, y \in V$. Let $\mathrm{Alg}^1(D)$ be the graph algebra of D with identity 1 adjoined. Example A.2 illustrates a graph D together with the corresponding Cayley multiplication table for $\mathrm{Alg}^1(D)$.



vertex	1	a	b	c	d	∞
1	1	a	b	c	d	∞
a	a	a	∞	∞	∞	∞
b	b	∞	∞	∞	∞	∞
c	c	∞	∞	c	c	∞
d	d	∞	∞	d	d	∞
∞	∞	∞	∞	∞	∞	∞

Figure A.2: A graph D with the corresponding Cayley table for $\mathrm{Alg}^1(D)$.

A *complete graph* contains all edges including loops. The complete graph of order n is denoted by K_n.

Definition 97 ([8], §1.6) A *regular expression* over an alphabet X is defined as follows:

(i) $\emptyset$ is a regular expression;
(ii) each member of X is a regular expression;
(iii) if x_1 and x_2 are regular expressions, then so is $(x_1 \cup x_2)$;
(iv) if x_1 and x_2 are regular expressions, then so is $(x_1 x_2)$; and
(v) if x is a regular expression, then so is x^*.

Lemma 98 ([8], Theorem 1.6) *Given an alphabet X, the regular languages over X are exactly the languages that are represented by regular expressions over X.*

A.2 Main algorithm

This section contains an algorithmic description of all regular languages recognised by graph algebras of finite graphs. The results in this appendix have been published in [35].

The following algorithm verifies whether a regular language is recognised by a graph algebra.

Algorithm

Input: A regular expression R defining a language L.

Output: Minimal graph algebra recognizing this language, if it exists.

Step 1. Find an automaton $\mathcal{A}$ recognizing L.

There are three major methods of transforming regular expressions into finite automata, which can be used in Step 1. The first method is due to Thompson [54], who used nondeterministic finite automata with λ-transitions. Berry and Sethi [4] gave an algorithm using nondeterministic finite automata. This method has been further developed by Brüggeman-Klein [9] and Chang, Paige [10]. The third method is due to Aho, Sethi and Ullman [2], and uses deterministic finite automata.

Step 2. Reduce $\mathcal{A}$ and find an equivalent minimal automaton $\mathcal{M}$.

The computation of a minimal automaton in Step 2 can be carried out in several ways. For instance, the *reduction algorithm* due to Moore starts with a given automaton and computes successive approximations of the Nerode equivalence (see, for example, [6]). A careful implementation of this algorithm has been proposed by Hopcroft [1], who proved that it can be carried out in time $O(N \log N)$ for N-state automaton.

Step 3. If $\mathcal{M}$ recognises a nontrivial language, then check whether $\mathcal{M}$ is of the form shown in Figures A.3, A.4, A.5, or A.6 up to notation of letters of the alphabet.

Step 4. If $\mathcal{M}$ is of the form in Figures A.3 A.4, A.5, or A.6, then the language L is recognised by the algebra $\mathrm{Alg}^1(D)$ of the graph D that consists of r copies of K_2, and $\ell - r$ copies of K_1 (when $n_1, \ldots, n_r \geq 1$ and $n_{r+1}, \ldots, n_\ell = 0$), together with at most one isolate.

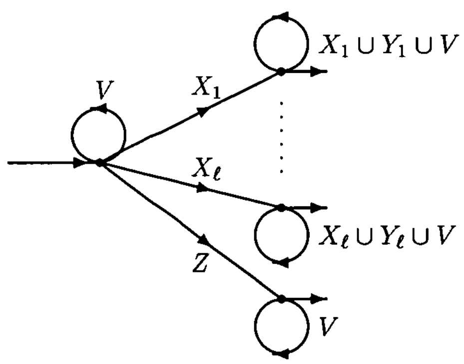


Figure A.3: Alphabet $X = X_1 \cup \ldots \cup X_\ell \cup Y_1 \cup \ldots \cup Y_\ell \cup Z \cup W$, where $X_i = \{x_{i1}, \ldots, x_{ik_i}\}$, $Y_i = \{y_{i1}, \ldots, y_{in_i}\}$, $Z = \{z_1, \ldots, z_p\}$, $W = \{w_1, \ldots, w_t\}$, $p, \ell, t, n_1, \ldots, n_\ell \geq 0$; $\ell + p \geq 1$; $k_1, \ldots, k_\ell \geq 1$.

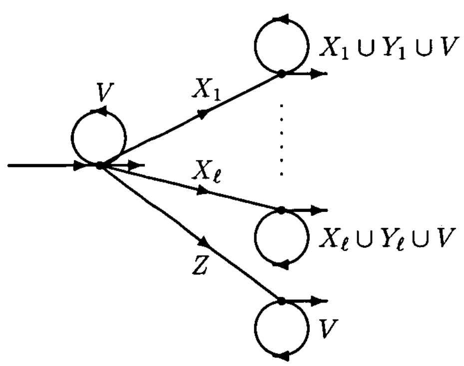


Figure A.4: Alphabet and all sets of letters as in Fig. A.3. If $p = 0$ and $\ell = 1$, then $n_1 \geq 1$.

A.3 Technical lemmas

Since we consider only finite graphs, it is clear that every language recognised by a graph algebra is regular. All regular languages can be characterised in terms of the syntactic monoid defined below.

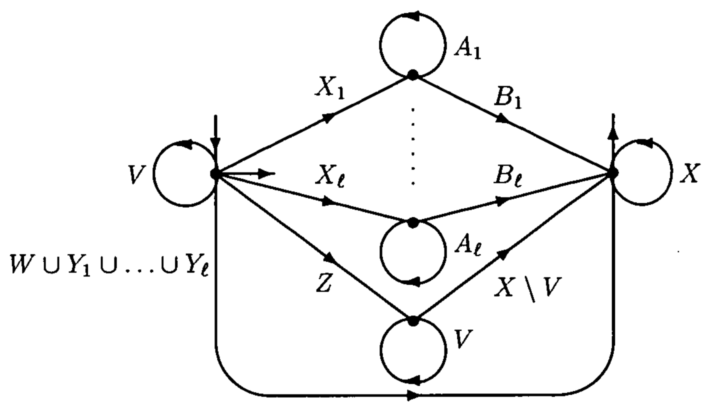


Figure A.5: Alphabet and all sets of letters as in Fig. A.3, and where $A_i = (X_i \cup Y_i \cup V)$ and $B_i = X \setminus A_i$, for $1 \leq i \leq \ell$.

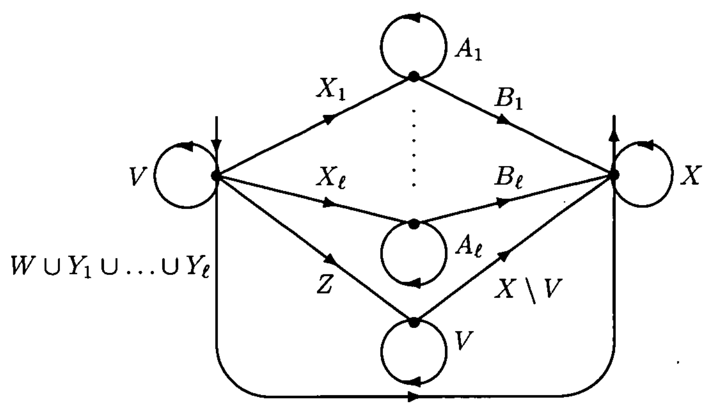


Figure A.6: Alphabet and all sets of letters as in Fig. A.3, and where $A_i = (X_i \cup Y_i \cup V)$ and $B_i = X \setminus A_i$, for $1 \leq i \leq \ell$. If $p = 0$ and $\ell = 1$, then $n_1 \geq 1$.

Definition 99 ([22], §3.1) For a subset $L \subseteq X^*$, the *syntactic monoid* $\mathrm{Syn}(L)$ of L is the quotient of X^* by σ_L, where

$$\begin{aligned} \sigma_L \;=\; & \{(x, y) \in X^* \times X^* : (\forall u, v \in X^*) \\ & uxv \in L \text{ if and only if } uyv \in L\}. \end{aligned}$$

Then we have

Lemma 100 ([14], 7.2.1) *For any subset L of X^* and any monoid M, the following conditions are equivalent:*

(i) *there exists a morphism $\phi : X^* \to M$ such that $L = \phi^{-1}(U)$ for some $U \subseteq M$;*

(ii) $\mathrm{Syn}(L)$ *divides* M.

The next result describes all graphs D such that $\mathrm{Alg}^1(D)$ is a syntactic monoid.

Lemma 101 ([37], Corollary 6) *Let D be a graph. Then the graph algebra $\mathrm{Alg}^1(D)$ is a syntactic monoid if and only if D has at most one isolated vertex and all other connected components of D are complete graphs with no more than two vertices.*

The following definition and results are also used to characterise syntactic monoids.

Definition 102 Let S be a monoid with a subset T, and let $s \in S$. Then the *context* of s with respect to T is denoted by $\mathrm{Cont}_T(s)$ and is defined by

$$\mathrm{Cont}_T(s) = \{(a, b) \in S^1 \times S^1 \mid asb \in T\}.$$

Lemma 103 ([14], Proposition 7.1.1) *Let $\phi : S \to T$ be a surjective morphism of semigroups, and let $A = \phi^{-1}(B)$ for some $B \subset T$. There exists an isomorphism $\psi : S/\sigma_A \to T/\sigma_B$.*

Lemma 104 ([14], Proposition 7.2.3) *A finite monoid S is syntactic if and only if there exists a finite alphabet X and a recognisable subset L of X^* such that $S \cong \mathrm{Syn}(L)$.*

Lemma 105 *A monoid S is syntactic if and only if S contains a subset T such that every two distinct elements of S have different contexts with respect to T.*

Proof. The 'only if' part. If S is syntactic, then $S \cong \mathrm{Syn}(L)$, for some finite alphabet X and subset $L \subseteq X^*$, by Lemma 104. Then every pair of elements of $\mathrm{Syn}(L)$ have distinct contexts with respect to $\sigma_L^\natural(L)$.

The 'if' part. Suppose that S contains a subset T such that every two distinct elements of S have different contexts with respect to T. Choose X so that there exists a surjective morphism $\phi : X^* \to S$, and let $L = \phi^{-1}(T)$. Then $\mathrm{Syn}(L) = X^*/\sigma_L \cong S/\sigma_T = S$, by Lemma 103. $\square$

A.4 Correctness of the algorithm

Our algorithm relies on the following theorem that characterises all languages recognised by graph algebras.

Theorem 106 *A language $L \subseteq X^*$ is recognised by the graph algebra of some graph if and only if the letters of X can be reordered and denoted by*

$$\begin{aligned} X &= \{x_{11}, \ldots, x_{1k_1}, \ldots, x_{\ell 1}, \ldots, x_{\ell k_\ell}, \\ &\quad y_{11}, \ldots, y_{1n_1}, \ldots, y_{\ell 1}, \ldots, y_{\ell n_\ell}, \\ &\quad z_1, \ldots, z_p, w_1, \ldots, w_t, v_1, \ldots, v_s\}, \end{aligned}$$

so that either L or $X^* \setminus L$ is represented by one of the the following regular expressions, where $V = v_1 + \cdots + v_s$, $X_i = x_{i1} + \cdots + x_{ik_i}$, $Y_i = y_{i1} + \cdots + y_{in_i}$, $Z = z_1 + \cdots + z_p$, and $\ell, p, s, t, n_1, \ldots, n_\ell, k_1, \ldots, k_\ell \geq 0$:

$$V^* \left(ZV^* + \sum_{i=1}^{\ell} X_i (X_i + Y_i + V)^* \right), \quad or \tag{A.1}$$

$$1 + V^* \left(1 + ZV^* + \sum_{i=1}^{\ell} X_i (X_i + Y_i + V)^* \right), \tag{A.2}$$

where if $p = 0$ and $\ell = 1$, then $n_1 \geq 1$.

Proof. The 'only if' part. Let $L \subseteq X^*$ be recognisable by $\mathrm{Alg}^1(D)$, for some graph D. Then there exists a morphism $\phi: X^* \to \mathrm{Alg}^1(D)$ and a subset T of $\mathrm{Alg}^1(D)$ such that $L = \phi^{-1}(T)$.

The image $\phi(X^*)$ is a submonoid of $\mathrm{Alg}^1(D)$. By Lemma 100, a language $L \subseteq X^*$ is recognisable by $\phi(X^*)$ if and only if $\mathrm{Syn}(L)$ divides $\phi(X^*)$. That is, there exists a submonoid S' of $\phi(X^*)$ and a morphism S' onto $\mathrm{Syn}(L)$.

A submonoid S' of $\phi(X^*)$ is induced by a subalgebra A of $\mathrm{Alg}(D)$. If A is generated by elements that belong to different connected components of D or one of them is ∞, then A is isomorphic to the graph algebra of D', where D' is a subgraph of D induced by the elements of $A \setminus \{\infty\}$. If the generators are taken from the same connected component of D, then A is a left zero semigroup. If we adjoin ∞ to a left zero semigroup, then we get a graph algebra, too.

The homomorphic image of a graph algebra (resp., a left zero semigroup) is a graph algebra (resp., a left zero semigroup), again. Thus if $L \subseteq X^*$ is recognisable by a graph algebra with identity adjoined, then the following two cases are possible:

Case 1: $\mathrm{Syn}(L)$ is isomorphic to $\mathrm{Alg}(G)$, for some graph G. Then Lemma 101 shows that G has at most one isolated vertex c_0, and all other connected components $C_1, \ldots, C_r$ and $C_{r+1}, \ldots, C_\ell$ of G are complete graphs with two or one vertices, respectively.

There exists a natural surjection $\sigma_L^\natural : X^* \to \mathrm{Alg}^1(G)$. Put $P = \sigma_L^\natural(L)$. It is clear that

$$\mathrm{Cont}_P(x) \neq \mathrm{Cont}_P(y), \quad \forall x \neq y \in \mathrm{Alg}^1(G), \tag{A.3}$$

as seen in Lemma 105.

First, assume that $\infty \notin P$ and $1 \notin P$. Then $\mathrm{Cont}_P(\infty) = \emptyset$. If G contains an isolate c_0 and $c_0 \notin P$, then $\mathrm{Cont}_P(c_0) = \emptyset$. This contradicts (A.3). Therefore $c_0 \in P$, if it exists, and $\mathrm{Cont}_P(c_0) = (1,1)$.

Similarly, $\mathrm{Cont}_P(C_i) = \emptyset$, for $r+1 \leq i \leq \ell$ if $C_i \notin P$. Therefore $C_i \in P$, for $r+1 \leq i \leq \ell$, and so $\mathrm{Cont}_P(C_i) = (1,1), (1,C_i), (C_i,1), (C_i,C_i)$.

Each connected component C_i contains two vertices, a, a', for $1 \leq i \leq r$. If $a, a' \notin P$, then $\mathrm{Cont}_P(a) = \mathrm{Cont}_P(a') = \emptyset$, a contradiction. If $a, a' \in P$, then $\mathrm{Cont}_P(a) = \mathrm{Cont}_P(a') = (1,1), (1,a), (a,1), (a,a), (1,a'), (a',1), (a',a'),$ $(a,a'), (a',a)$, again a contradiction. Therefore precisely one of a, a', say a belongs to P. Then $\mathrm{Cont}_P(a) = (1,1), (1,a), (a,1), (a,a), (1,a'), (a,a')$ and $\mathrm{Cont}_P(a') = (a,1), (a,a), (a,a')$.

Since $(1,1) \notin \mathrm{Cont}_P(1)$, we see that $\mathrm{Cont}_P(1) \neq \mathrm{Cont}_P(c_0)$, and also that $\mathrm{Cont}_P(1) \neq \mathrm{Cont}_P(a)$, for all $a \in C_i$ where $a \in P$. If C_i contains two vertices a, a' and $a' \notin P$, then $(1,a) \in \mathrm{Cont}_P(1) \setminus \mathrm{Cont}_P(a')$. Therefore all elements of $\mathrm{Alg}^1(G)$ have unique contexts.

Denote by $z_1, \ldots, z_p$ the elements of X that are mapped by ϕ to c_0, denote by $w_1, \ldots, w_t$ the elements that are mapped to ∞, by $v_1, \ldots, v_s$ the elements that are mapped to 1, and by $x_{i1}, \ldots, x_{ik_i}$ the elements that go to the only vertex of C_i, for $i = r+1, \ldots, \ell$. Further, for $i = 1, \ldots, r$, let $x_{i1}, \ldots, x_{ik_i}$ be the elements of X that are taken to the vertex of C_i chosen in P, and let $y_{i1}, \ldots, y_{in_i}$ be the elements which are taken to the other vertex of C_i. It is routine to verify that then L is represented by the regular expression of the form (A.1).

Second, assume that $\infty \notin P$ and $1 \in P$. The same argument shows that P consists of the identity element, the only isolated vertex c_0, if it exists, and just one vertex from each other connected component of G. Therefore L is represented by the regular expression of the form (A.2).

It is worth mentioning that in this case $G \neq K_1$. Indeed, suppose that $G = K_1$, and let $V(G) = \{a\}$. Then $\mathrm{Cont}_P(a) = \mathrm{Cont}_P(1)$ if $a \in P$, and $\mathrm{Cont}_P(a) = \mathrm{Cont}_P(\infty)$ if $a \notin P$, a contradiction.

Third, assume that $\infty \in P$ and $1 \in P$. In this case P contains just one vertex from each connected component isomorphic to K_2. Relabelling letters of the alphabet we obtain the complement of the set P considered above, in the case where $1 \notin P$ and $\infty \notin P$. Since $(\sigma_L^\natural)^{-1}P = L$, it follows that $(\sigma_L^\natural)^{-1}(\mathrm{Syn}(L) \setminus P) = X^* \setminus L$. Therefore $X^* \setminus L$ is also recognised by $\mathrm{Syn}(L)$. Hence $X^* \setminus L$ is represented by the regular expression (A.1).

Finally, assume that $\infty \in P$ and $1 \notin P$. Then $G \neq K_1$ and $X^* \setminus L$ is represented by the regular expression (A.2).

Case 2: $\mathrm{Syn}(L)$ is a left zero semigroup with identity adjoined. If a left zero semigroup has more than two elements, then for every subset T of $\mathrm{Syn}(L)$ there exist two elements with the same context with respect to this subset. This contradicts Lemma 105 and shows that $|\,\mathrm{Syn}(L)| \leq 3$.

There exists a natural surjection $\sigma_L^\natural : X^* \to \mathrm{Syn}(L)$, where $L = \phi^{-1}(P)$, for some $P \subset \mathrm{Syn}(L)$.

First, assume that $|\,\mathrm{Syn}(L)| = 1$. Then $X^* = \phi^{-1}(1)$, and so $L = X^*$. This is a special case of (A.1), when $p = 0, \ell = 0, k_0 = 0$ and $t = 0$.

Second, assume that $|\,\mathrm{Syn}(L)| = 2$, and let $\mathrm{Syn}(L) = \{a, 1\}$. Denote by $x_1, \ldots, x_k$ the elements of X that are mapped to a by ϕ, and by $v_1, \ldots, v_s$ the elements of X that are mapped to 1. If $P = \{a, 1\}$, then $\mathrm{Cont}_P(1) = \mathrm{Cont}_P(a)$, which contradicts Lemma 105. If $P = \{1\}$, then L is represented by the expression $(v_1 + \cdots + v_s)^*$ of type (A.2), where $p = 0, \ell = 1, k_1 = k, n_1 = 0$ and $t = 0$. If $P = \{a\}$, then L is represented by

$$(v_1 + \cdots + v_s)^*(x_1 + \cdots + x_k)X^*,$$

a special case of (A.1), where $p = 0, \ell = 1, k_1 = k, n_1 = 0$ and $t = 0$.

Third, assume that $|\,\mathrm{Syn}(L)| = 3$, and let $\mathrm{Syn}(L) = \{a, b, 1\}$. Denote by $x_1, \ldots, x_k$ the elements of X that are mapped to a by ϕ, by $y_1, \ldots, y_m$ the elements of X that are mapped to b by ϕ, and by $v_1, \dots, v_s$ the elements of X that are mapped to 1.

If $P = \{1\}$, $P = \{a, b\}$ or $P = \{a, b, 1\}$, then $\mathrm{Syn}(L)$ contains two elements with the same context with respect to P. If $P = \{a\}$ (the case where $P = \{b\}$ is dual), then L is represented by

$$(v_1 + \cdots + v_s)^*(x_1 + \cdots + x_k)X^*.$$

That is, L is given by an expression of the form (A.1), where $p = 0$, $\ell = 1$, $k_1 = k, n_1 = m$ and $t = 0$. If $P = \{1, a\}$ (the case where $P = \{1, b\}$ is dual), then L is represented by

$$1 + (v_1 + \cdots + v_s)^*(1 + (x_1 + \cdots + x_k))X^*,$$

an expression of type (A.2), where $p = 0, \ell = 1, k_1 = k, n_1 = m$ and $t = 0$.

The 'if' part. Suppose that L is given by the regular expressions (A.1) or (A.2), where $n_1, \dots, n_r \geq 1$ and $n_{r+1}, \dots, n_\ell = 0$. Consider the graph D which consists of an isolated vertex c_0 (if $p > 0$), r copies of K_2, and $\ell - r$ copies of K_1. By Lemma 101, $\mathrm{Alg}^1(D)$ is a syntactic monoid that recognises some language. When L is a regular expression of the form (A.1), let $T \subseteq \mathrm{Alg}^1(D)$ be the set which comprises the only isolated vertex c_0, if it exists, and one vertex from each connected component of D. If L is given by the regular expressions (A.2), then we also include 1 in T.

By Lemma [22], 1.7.3, there exists a unique morphism $\phi: X^* \to \mathrm{Alg}^1(D)$, which maps $z_1, \dots, z_p$ to c_0, the letters $w_1, \dots, w_t$ to ∞, and $v_1, \dots, v_s$ to 1; and, for $i = r+1, \dots, \ell$, maps $x_{i1}, \dots, x_{ik_i}$ to the vertex of the i-th copy C_i of K_1; and, for $i = 1, \dots, r$, maps $x_{i1}, \dots, x_{ik_i}$ to the vertex C_i of the i-th copy of K_2 which belongs to T; and, finally, for $i = 1, \dots, r$, maps $y_{i1}, \dots, y_{in_i}$ to the other vertex of the i-th copy of K_2.

We will show that $L = \phi^{-1}(T)$. Take any $u \in L$. By the definition of (A.1) and (A.2), $\phi(u) \in T$. Therefore $L \subseteq \phi^{-1}(T)$.

On the other hand, let $\phi(u) \in T$, for some $u = a_1 \cdots a_k \in X^*$, where $a_i \in X$. If $1 \in T$ and $\phi(u) = 1$, then $\phi(a_i) = 1$, for all $1 \leq i \leq k$. Then $a_i \in V \cup \{1\}$, and so $u \in 1 + V^* \subseteq L$. If $\phi(u) = c_0$ and $1 \in T$ (resp., $1 \notin T$), then there exists $1 \leq j \leq k$ such that $\phi(a_j) = c_0$. Then $\phi(a_i) = 1$, for all $1 \leq i \leq k$ and $i \neq j$. Thus $u \in 1 + V^*(1 + ZV^*) \subseteq L$ (resp., $u \in V^*ZV^* \subseteq L$). If $\phi(u) = C_i$ and $1 \in T$ (resp., $1 \notin T$), then there exist $1 \leq j \leq k$ such that $\phi(a_j) = C_i$ and $\phi(a_h) \neq C_i$, for $h < j$. Then $\phi(a_h) = 1$, and so $a_h \in V \cup \{1\}$

(resp., $a_h \in V$), for $h < j$. For $g > h$, we see that either a_g belongs to the same connected component as a_h, or $a_g \in V \cup \{1\}$ (resp., $a_g \in V$). Therefore $u \in 1 + V^*(1 + V^*X_i(X_i + Y_i + V)^*) \subseteq L$ (resp., $u \in V^*X_i(X_i + Y_i + V)^* \subseteq L$). Thus in every case $u \in L$, and so $\phi^{-1}(T) \subseteq L$. Therefore $\phi^{-1}(T) = L$, and by Definition 95, L is recognised by $\mathrm{Alg}^1(D)$.

Suppose that $X^* \setminus L$ is represented by the regular expression (A.1) or (A.2). The preceding arguments shows that $X^* \setminus L$ is recognised by a finite graph algebra $\mathrm{Alg}^1(D)$, where $\psi : X^* \to \mathrm{Alg}^1(D)$ and $\psi^{-1}(T) = X^* \setminus L$, for some surjective morphism ψ and subset T of $\mathrm{Alg}^1(D)$. Then L is recognised by $\mathrm{Alg}^1(D)$ as $\psi^{-1}(\mathrm{Alg}^1(D) \setminus T) = L$. □

Thus if a language L is defined by a regular expression of type (A.1) (resp., (A.2)), then it recognised by the automaton in Figure A.3 (resp., Figure A.4). Similarly, if the complement $X^* \setminus L$ is represented by the expression of type (A.1) (resp., (A.2)), then the automaton shown in Figure A.5 (resp., Figure A.6) recognises L.

Example 107 Suppose that $X = \{\alpha, \beta, \gamma\}$, and $L = \alpha^*\beta\alpha^*$.

The minimal automaton **M** which recognises L is given by:

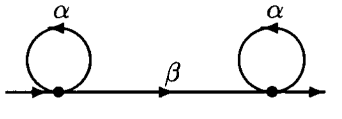


Figure A.7: The state diagram for **M**.

The state diagram in Figure (A.7) corresponds to the state diagram in Figure (A.3). Therefore L is recognised by the graph algebra $\mathrm{Alg}^1(D)$ where D consists of one isolate c_0.

The alphabet X can be relabelled according to Theorem 106, namely, $\alpha = v_1, \beta = z_1, \gamma = w_1$. The morphism $\phi : X^* \to \mathrm{Alg}^1(D)$ is given by

$$\alpha \to 1; \quad \beta \to c_0; \quad \gamma \to \infty.$$

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Glossary of Notation

Symbol	Definition	Pages
$\lvert a\rvert$	the order of an element a	5
A^+	the free semigroup over an alphabet A	13
A^*	the free monoid over an alphabet A	13
$\mathrm{Alg}^1(D)$	the graph algebra associated with a graph D	94
$\gcd(a,b)$	the greatest common divisor of a and b	19
$\langle A\rangle$	the subsemigroup generated by A	6
$\mathcal{A}(S,\Lambda,p,d,H)$	an automaton	92
$\mathrm{Ann}(S)$	the annihilator graph of a semigroup S	77
A_∞	an infinite ascending chain	14
B_2	the five element Brandt semigroup	29
$C(G)$	the centre of a group G	5
$\mathrm{Cay}(S,T)$	the Cayley graph of S with respect to $T\subseteq S$	35
$\mathrm{Cont}_T(s)$	the context of $s\in S$ with respect to $T\subseteq S$.	98
$\mathcal{D}$	Green's relation	8
$\mathcal{D}^S$	Green's relation in S	9
$\mathrm{Div}(S)$	the divisibility graph of S	49
$\mathrm{Div}(U)^S$	the subgraph of $\mathrm{Div}(S)$ induced by $U\subseteq S$	49
D_∞	an infinite descending chain	14
$E(G)$	the edge set in a graph G	13
e_g	the identity element of a group G, where $g\in G$	4
$G_1\times\ldots\times G_n$	the direct product of a finite set of (semi)groups	5
$G(V,E)$	a graph with vertex set V and edge set E	13

$G_{i\lambda}$	an $\mathcal{H}$-class of $M^0(G; I, \Lambda; P)$	10
$G^j_{i\lambda}$	an $\mathcal{H}$-class of $\mathcal{M}^0(G_j; I_j, \Lambda_j; P_j)$	67
G_{i*}	an $\mathcal{R}$-class of $M^0(G; I, \Lambda; P)$	10
G^j_{i*}	an $\mathcal{R}$-class of $M^0(G_j; I_j, \Lambda_j; P_j)$	67
$G_{*\lambda}$	an $\mathcal{L}$-class of $M^0(G; I, \Lambda; P)$	10
$G^j_{*\lambda}$	an $\mathcal{L}$-class of $M^0(G_j; I_j, \Lambda_j; P_j)$	67
$g_{i\lambda}$	an element in $G_{i\lambda}$	57
$g^j_{i\lambda}$	an element in $G^j_{i\lambda}$	67
$GL_j(K)$	the group of $j \times j$ invertible matrices over a field K	12
G_p	the p-primary component of a group G	5
G_y	the maximal subgroup of S_y	12
$\mathcal{H}$	Green's relation	8
$\mathcal{H}^S$	Green's relation in S	9
$(h; i, \lambda)$	an element of $M^0(H; I, \Lambda; P)$	10
$\mathcal{J}$	Green's relation	8
$\mathcal{J}^S$	Green's relation in S	9
J_a	the $\mathcal{J}$-class containing a	9
$J(a)$	the principal ideal generated by a	9
K	a skew field	12
K_n	the complete undirected graph of order n	94
K_∞	the complete symmetric graph of infinite order	13
$\mathcal{L}$	Green's relation	8
$\mathcal{L}^S$	Green's relation in S	9
$L(\mathbf{A})$	the language recognised by $\mathbf{A}$	92

L_a	the $\mathcal{L}$-class containing a	9
L_a^U	the $\mathcal{L}^U$-class containing a	9
M_n	the semigroup $M_n(G)$ or $M_n(K)$	27
$M_n(K)$	the set of all $n \times n$ linear matrices over a field K	12
$M_n(G)$	the set of all $n \times n$ monomial matrices over G	12
$\mathcal{M}(G; I, \Lambda; P)$	the Rees matrix semigroup over G	10
$\mathcal{M}^0(G; I, \Lambda; P)$	the Rees matrix semigroup over G^0	10
$\pi(G)$	the projection of G	20
$p_{\lambda i}$	an entry of a sandwich matrix	9
$\phi(m)$	the Euler phi function for m	32
$\mathrm{Pow}(S)$	the power graph of a semigroup S	15
$\mathrm{Pow}(U)^S$	the subgraph of $\mathrm{Pow}(S)$ induced by $U \subseteq S$	16
$\mathbb{Q}$	the set of rational numbers	56
$\mathbb{R}$	the set of real numbers	56
$\mathcal{R}$	Green's relation	8
$\mathcal{R}^S$	Green's relation in S	9
R_a	the $\mathcal{R}$-class containing a	9
R_a^T	the $\mathcal{R}^T$-class containing a	9
$R(a)$	the principal right ideal generated by a	9
R^c	the smallest compatible relation containing R	7
R^e	the smallest equivalence relation generated by R	7
R^∞	the transitive closure of R	7
$R^\#$	the smallest congruence generated by R	7
$\rho^\natural$	the natural map from S to S/ρ	11
$\lvert S\rvert$	the cardinality of a set S	5

S/N	the Rees quotient of S by N	5
S_y	an Archimedean semigroup of a semilattice Y	11
$\mathrm{Syn}(L)$	the syntactic monoid of a language L	97
S^0	the semigroup S with adjoined zero	6
S^1	the semigroup S with adjoined identity	6
σ_L	the syntactic congruence of a set $L \subseteq X^*$	97
$T^\star$	the set $\{uv \mid u, v \in T, u \neq v\}$	79
(u, v)	an edge in a graph	14
$V(G)$	the vertex set in a graph G	13
Y	a semilattice	11
$\mathbb{Z}^+$	the set of positive integers	14
$\mathbb{Z}_p$	the cyclic group or order p	5
$\mathbb{Z}_{p^\infty}$	the quasicyclic p-group	5
1_X	the equality relation of a set X	6

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